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Symmetric Ternary Interpolating C^1 Subdivision Scheme

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Abstract

A ternary 4-point interpolating subdivision scheme is proposed that generates the limiting curve of C^1 continuity. Performance of the proposed subdivision scheme is improved using a tension parameter. The improved subdivision scheme generates a family of C^1 limiting curves for certain range of tension parameter. Laurent polynomial method is used to investigate the derivative continuity of the subdivision schemes. The role of the tension parameter is exposed in two examples.

Keywords: Ternary; Interpolating subdivision scheme; Convergence and smoothness; Mask; Laurent polynomial

1 Introduction

Significance role of subdivision schemes can not be denied due to the wide applications in computer aided geometric design, computer graphics, geometric modelling, medical surgery simulations and reverse engineering. Curve subdivision schemes can be classified into two important branches, interpolating and approximating. Interpolating subdivision scheme play a vital role in the field of subdivision because it preserve the data obtained at the former stage and refines new data by inserting values corresponding to intermediate points. Siddiqi and Rehan [6] introduced a new corner cutting subdivision scheme that generates limiting curve of C^1 continuity. Hassan and Dodgson [2] developed a ternary 3-point approximating subdivision scheme that generates C^2 limiting curve. Siddiqi and Rehan [8] introduced a ternary 3-point approximating scheme that generates C^2 limiting curve. Siddiqi and Rehan [7] also modified the ternary 3-point approximating subdivision scheme that generates a family of $C^n,\ n = 1, 2$ limiting curves for a certain range of tension parameter. Hassan and Dodgson [2] presented a ternary 3-point interpolating subdivision scheme that generates a family of C^1 limiting curves. Zheng *et al.* [9] also developed a ternary 3-point interpolating subdivision scheme that generates C^1 limiting curve. Zheng *et al.* [10] analyzed the fractal property of the ternary 3-point interpolating subdivision scheme [2] with two parameters.

Kwan *et al.* [5] presented a ternary 4-point approximating subdivision scheme that generates C^2 limiting curve. Hassan *et al.* [3] proposed a ternary 4-point interpolating subdivision scheme that generates C^2 limiting curves for a certain range of tension parameter. Beccari *et al.* [1] introduced a non-stationary ternary 4-point interpolating subdivision scheme that generates C^2 limiting curves using tension parameter. Huawei and Kaihuai [4] developed an improved form of ternary 4-point interpolating subdivision scheme that generates limiting curve of C^2 continuity. Zheng *et al.* [11] investigated the differentiable properties of the ternary 4-point interpolating subdivision scheme [3].

A subdivision algorithm recursively refines the initial polygon to produce a sequence of finer polygons that converge to a smooth limiting curve. Each subdivision scheme is associated with a mask $a = \{a_i\},\ i \in Z$. The ternary subdivision scheme is the process which recursively define a sequence of control points $f^k = f_i^k,\ i \in Z$ by the rule of the form with mask $a = \{a_i\},\ i \in Z$

$$f_i^{k+1} = \sum_{j \in Z} a_{i-3j} f_j^k,$$

which is formally denoted by $f^{k+1} = Sf^k = S^k f^0$. A subdivision scheme is said to be uniformly convergent if for every initial data $f^0 = \{f_i\},\ i \in Z$, there is a continuous function f such that for any closed interval $[a, b]$

$$\lim_{k \longrightarrow \infty} \sup_{i \in Z \cap 3^k [a,b]} |f_i^k - f(3^{-k} i)| = 0.$$

Obviously $f = S^\infty f^0$ is considered to be a limit function of subdivision scheme S.

In this paper, ternary 4-point interpolating subdivision scheme is introduced using the basis functions of cubic Catmull Rom spline. The scheme is defined as

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= af_{i-1}^k + bf_i^k + cf_{i+1}^k + df_{i+2}^k, \\ f_{3i+2}^{k+1} &= df_{i-1}^k + cf_i^k + bf_{i+1}^k + af_{i+2}^k. \end{aligned}$$

where f_i^0 is a set of initial control points with the mask $a = \frac{1}{2}(-w + 2w^2 - w^3)$, $b = \frac{1}{2}(2 - 5w^2 + 3w^3)$, $c = \frac{1}{2}(w + 4w^2 - 3w^3)$ and $d = \frac{1}{2}(-w^2 + w^3)$. With the choice $w = 1/3$, the scheme can be written as

$$f_{3i}^{k+1} = f_i^k,$$

$$\begin{aligned} f_{3i+1}^{k+1} &= \frac{-2}{27} f_{i-1}^k + \frac{21}{27} f_i^k + \frac{9}{27} f_{i+1}^k - \frac{1}{27} f_{i+2}^k, \\ f_{3i+2}^{k+1} &= \frac{-1}{27} f_{i-1}^k + \frac{9}{27} f_i^k + \frac{21}{27} f_{i+1}^k - \frac{2}{27} f_{i+2}^k. \end{aligned} \quad (1.1\ )$$

This paper is organized as follows. In section 2, analysis of the ternary 4-point interpolating subdivision scheme is presented. Analysis of an improved ternary 4-point interpolating subdivision scheme is shown in section 3. In section 4, two examples are considered to demonstrate the role of the tension parameter. Conclusion is drawn in section 5.

2 Analysis of ternary subdivision scheme

For the convergent subdivision scheme S, the corresponding mask $\{a_i\},\ i \in Z$ necessarily satisfies

$$\sum_{j\in Z} a_{3j} = \sum_{j\in Z} a_{3j+1} = \sum_{j\in Z} a_{3j+2} = 1. \quad (2.1\ )$$

Introducing a symbol called the Laurent polynomial

$$a(z) = \sum_{i\in Z} a_i z^i$$

of a mask $\{a_i\},\ i \in Z$ with finite support. The corresponding symbols play an efficient role to analyze the convergence and smoothness of subdivision scheme. With the symbol, Hassan *et al.* [3] provided a sufficient and necessary condition for a uniform convergent subdivision scheme. A subdivision scheme S is uniform convergent if and only if there is an integer $L \geq 1$, such that

$$\left\| (\frac{1}{3} S_1)^L \right\|_\infty < 1.$$

The subdivision scheme S_1 with symbol $a_1(z)$ is related to S with symbol $a(z)$, where $a_1(z) = \frac{3z^2}{1+z+z^2} a(z)$. The subdivision scheme S with symbol $a(z)$ satisfying equation (2.1) then there exists a subdivision scheme S_1 with the property

$$df^k = S_1 df^{k-1},\ k = 1, 2, ...,$$

where $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 3^k(f_{i+1}^k - f_i^k) : i \in Z\}$. The norm $\|S\|_\infty$ of a subdivision scheme S with a mask $\{a_i\},\ i \in Z$ is defined by

$$\|S\|_\infty = max \left\{ \sum_{i\in Z} |a_{3i}|, \sum_{i\in Z} |a_{3i+1}|, \sum_{i\in Z} |a_{3i+2}| \right\}.$$

Theorem 1.
Ternary 4-point interpolating subdivision scheme defined in the section 1 converges and has smoothness C^1.
Proof.
Consider the refinement rules of the ternary 4-point interpolating subdivision scheme defined in equation (1.1) and the Laurent polynomial $a(z)$ for the mask of the subdivision scheme can be written as

$$a(z) = \frac{-1}{27}z^{-5} - \frac{2}{27}z^{-4} + \frac{9}{27}z^{-2} + \frac{21}{27}z^{-1} + 1 + \frac{21}{27}z^{1} + \frac{9}{27}z^{2} - \frac{2}{27}z^{4} - \frac{1}{27}z^{5}.$$

Laurent polynomial method is used to prove the smoothness of the ternary 4-point interpolating subdivision scheme to be C^1. Taking

$$b^{[m,L]}(z) \;=\; \frac{1}{3^L}a_m^{[L]}(z), \quad L = 1, 2, , 3, ..., m,$$

where

$$a_m(z) \;=\; \left(\frac{3z^2}{1+z+z^2}\right)a_{m-1}(z) = \left(\frac{3z^2}{1+z+z^2}\right)^m a(z)$$

and

$$a_m^{[L]}(z) = \prod_{j=0}^{L-1} a_m(z^{3^j}).$$

With a choice of $m = 1$ and $L = 1$, it can be written as

$$b^{[1,1]}(z) = \frac{1}{3^1}a_1(z) = \frac{-1}{27}z^{-3} - \frac{1}{27}z^{-2} + \frac{2}{27}z^{-1} + \frac{8}{27} + \frac{11}{27}z^{1} + \frac{8}{27}z^{2} + \frac{2}{27}z^{3} - \frac{1}{27}z^{4} - \frac{1}{27}z^{5}.$$

The norm of subdivision scheme $\frac{1}{3}S_1$ is

$$\left\|\frac{1}{3}S_1\right\|_\infty = max\left\{\sum_\beta |b^{[1,1]}_{\gamma+3\beta}| : \gamma = 0, 1, 2\right\} = max\left\{\frac{11}{27}, \frac{13}{27}, \frac{11}{27}\right\} = \frac{13}{27} < 1,$$

therefore the subdivision scheme S is convergent.
In order to prove the ternary 4-point interpolating subdivision scheme to be C^1. Considering $m = 2$ and $L = 1$, the Laurent polynomial gives

$$b^{[2,1]}(z) = \frac{1}{3^1}a_2(z) = \frac{-1}{9}z^{-1} + \frac{1}{3}z^{1} + \frac{5}{9}z^{2} + \frac{1}{3}z^{3} - \frac{1}{9}z^{5}.$$

The norm of subdivision $\frac{1}{3}S_2$ is

$$\left\|\frac{1}{3}S_2\right\|_\infty \;=\; max\left\{\sum_\beta |b^{[2,1]}_{\gamma+3\beta}| : \gamma = 0, 1, 2\right\} = max\left\{\frac{1}{3}, \frac{1}{3}, \frac{7}{9}\right\} = \frac{7}{9} < 1,$$

therefore the subdivision scheme S_1 is convergent and $S \in C^1$.
In order to prove the ternary 4-point interpolating subdivision scheme to be C^2. Considering $m = 3$ and $L = 1$, the Laurent polynomial gives

$$b^{[3,1]}(z) = \frac{1}{3^1}a_3(z) = \frac{-1}{3}z^1 + \frac{1}{3}z^2 + z^3 + \frac{1}{3}z^4 - \frac{1}{3}z^5.$$

The norm of subdivision $\frac{1}{3}S_3$ is

$$\left\|\frac{1}{3}S_3\right\|_\infty = max\left\{\sum_\beta |b^{[3,1]}_{\gamma+3\beta}| : \gamma = 0,1,2\right\} = max\left\{1, \frac{2}{3}, \frac{2}{3}\right\} = 1,$$

therefore consider $m = 3$ and $L = 2, 3$ this gives

$$\left\|(\frac{1}{3}S_3)^2\right\|_\infty \geq 1 \quad and \quad \left\|(\frac{1}{3}S_3)^3\right\|_\infty \geq 1.$$

Consequently, this shows that the ternary 4-point interpolating subdivision scheme is not C^m continuous for $m > 2$.

3 Improved ternary 4-point interpolating subdivision scheme

Ternary 4-point interpolating subdivision scheme is introduced as defined in equation (1.1) that generates the limiting curve of C^1 continuity. The limitation of the subdivision scheme is removed by introducing a tension parameter μ that generates a family of C^1 limiting curves. The subdivision scheme is defined as

$$\begin{aligned} f^{k+1}_{3i} &= f^k_i, \\ f^{k+1}_{3i+1} &= (\frac{-2}{27} - \mu)f^k_{i-1} + (\frac{21}{27} + \mu)f^k_i + (\frac{9}{27} + \mu)f^k_{i+1} + (\frac{-1}{27} - \mu)f^k_{i+2}, \\ f^{k+1}_{3i+2} &= (\frac{-1}{27} - \mu)f^k_{i-1} + (\frac{9}{27} + \mu)f^k_i + (\frac{21}{27} + \mu)f^k_{i+1} + (\frac{-2}{27} - \mu)f^k_{i+2} \quad (3.1\ ) \end{aligned}$$

Theorem 2.
Improved ternary 4-point interpolating subdivision scheme defined in equation (3.1) generates a family of C^1 limiting curves for the range of tension parameter $\mu \in]\frac{-1}{18}\ \frac{1}{9}[$.
Proof.
Consider the refinement rules of an improved ternary 4-point interpolating subdivision scheme defined in equation (3.1) and the Laurent polynomial $a(z)$ for the mask of the subdivision scheme can be written as

$$\begin{aligned} a(z) &= (\frac{-1}{27} - \mu)z^{-5} + (\frac{-2}{27} - \mu)z^{-4} + (\frac{9}{27} + \mu)z^{-2} + (\frac{21}{27} + \mu)z^{-1} + 1 + (\frac{21}{27} + \mu)z^1 \\ &\quad + (\frac{9}{27} + \mu)z^2 + (\frac{-2}{27} - \mu)z^4 + (\frac{-1}{27} - \mu)z^5. \end{aligned}$$

Laurent polynomial method is used, as defined in section 2, to prove the smoothness of the improved ternary 4-point interpolating subdivision scheme to be C^1. With a choice of $m = 1$ and $L = 1$, it can be written as

$$\begin{aligned} b^{[1,1]}(z) = \frac{1}{3^1} a_1(z) \quad &= \quad (\frac{-1}{27} - \mu) z^{-3} - \frac{1}{27} z^{-2} + (\frac{2}{27} + \mu) z^{-1} + \frac{8}{27} + \frac{11}{27} z^1 + \frac{8}{27} z^2 \\ &+ (\frac{2}{27} + \mu) z^3 - \frac{1}{27} z^4 + (\frac{-1}{27} - \mu) z^5. \end{aligned}$$

The norm of subdivision scheme $\frac{1}{3} S_1$ is

$$\begin{aligned} \left\| \frac{1}{3} S_1 \right\|_\infty &= max \left\{ \sum_\beta |b^{[1,1]}_{\gamma+3\beta}| : \gamma = 0, 1, 2 \right\} \\ &= max \left\{ |\frac{10}{27} + \mu| + |\frac{-1}{27} - \mu|, \frac{13}{27}, |\frac{10}{27} + \mu| + |\frac{-1}{27} - \mu| \right\} < 1, \end{aligned}$$

therefore the subdivision scheme S is convergent for the range of tension parameter $\mu \in]\frac{-19}{27}\ \frac{8}{27}[$.
In order to prove the improved ternary 4-point interpolating subdivision scheme to be C^1. Considering $m = 2$ and $L = 1$, the Laurent polynomial gives

$$\begin{aligned} b^{[2,1]}(z) = \frac{1}{3^1} a_2(z) \quad &= \quad (\frac{-1}{9} - 3\mu) z^{-1} + (3\mu) + (\frac{1}{3} + 3\mu) z^1 + (\frac{5}{9} - 6\mu) z^2 \\ &+ (\frac{1}{3} + 3\mu) z^3 + (3\mu) z^4 + (\frac{-1}{9} - 3\mu) z^5. \end{aligned}$$

The norm of subdivision scheme $\frac{1}{3} S_2$ is

$$\begin{aligned} \left\| \frac{1}{3} S_2 \right\|_\infty &= max \left\{ \sum_\beta |b^{[2,1]}_{\gamma+3\beta}| : \gamma = 0, 1, 2 \right\} \\ &= max \left\{ |3\mu| + |\frac{1}{3} + 3\mu|, |3\mu| + |\frac{1}{3} + 3\mu|, |\frac{5}{9} - 6\mu| + 2|\frac{-1}{9} - 3\mu| \right\} < 1, \end{aligned}$$

therefore the subdivision scheme S_1 is convergent and $S \in C^1$ for the range of tension parameter $\mu \in]\frac{-1}{18}\ \frac{1}{9}[$.

4 Examples

Two examples have been demonstrated to show the role of the tension parameter μ and the limiting curve passes through the discrete set of data points as shown in Figure 1. The limiting curve tends to shrink towards control polygon with the decrease of value of the tension parameter $\mu = 0.02, 0, -0.02$.

5 Conclusion

A ternary 4-point interpolating subdivision scheme has been introduced using Catmull Rom spline basis functions that generates C^1 limiting curve. The proposed subdivision scheme has also been improved using a tension parameter μ. Improved subdivision scheme generates a family of C^1 limiting curves. Laurent polynomial method has been used to analysis the smoothness of the subdivision schemes. Examples have been considered to demonstrate the role of the tension parameter that provides more control to the geometric designers.

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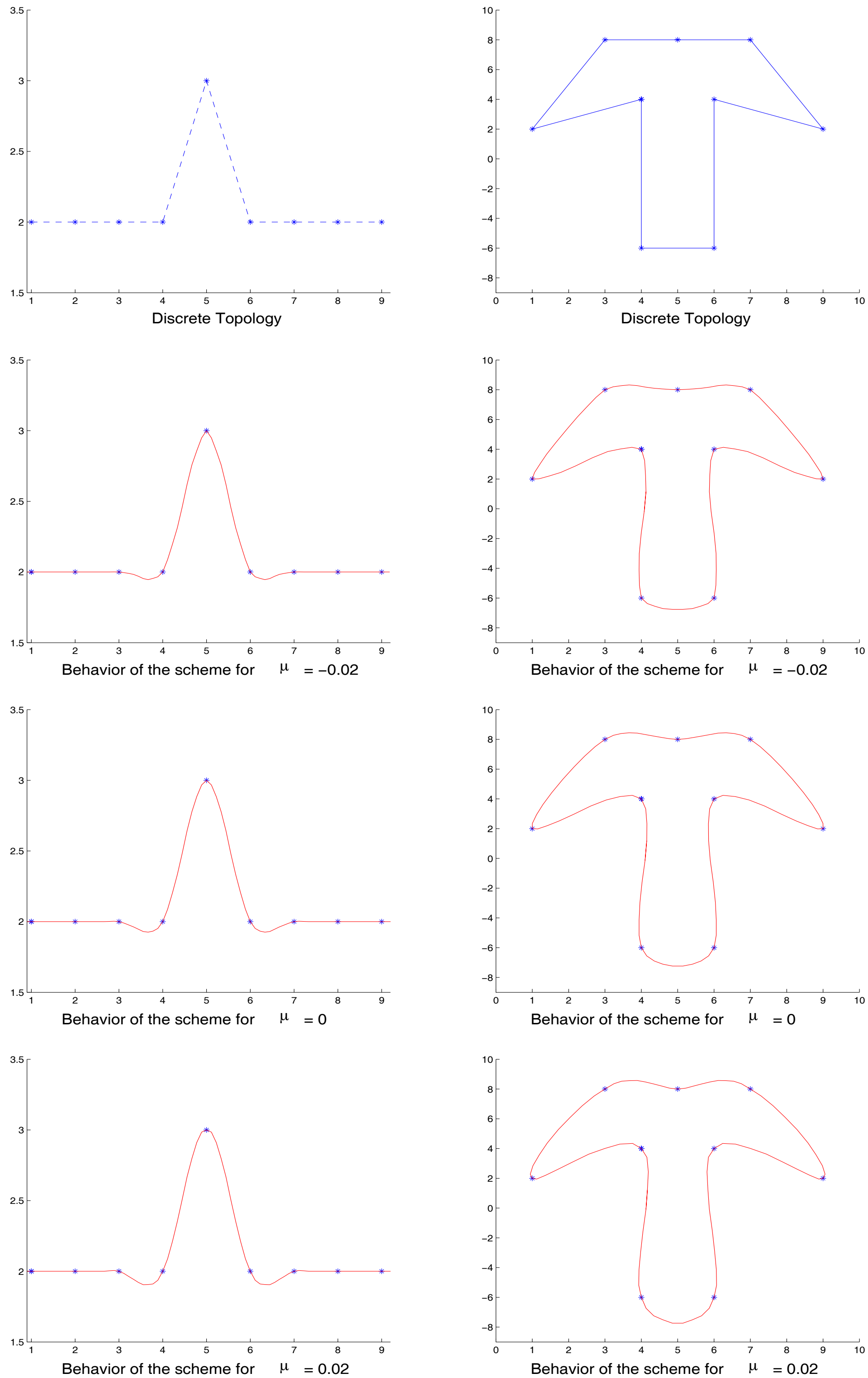

Figure 1: Behaviour of an improved ternary 4-point interpolating subdivision scheme for different values of the global tension parameter.