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Remarks on E-Order-Preserving Transformation Semigroups

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Abstract

Let $T(X)$ be the full transformation semigroup on a set X. For a partially ordered set X, let E be an arbitrary equivalence relation on X. We consider a subsemigroup of $T(X)$ defined by

$$T_{EO}(X) =: \{\alpha \in T(X) : \forall\ x, y \in X, (x, y) \in E, x \leq y \Rightarrow (x\alpha, y\alpha) \in E, x\alpha \leq y\alpha\}$$

and call it the *E-order-preserving transformation semigroup on* X. The purpose of this paper is to investigate relationships between $T_{EO}(X)$ and some subsemigroups of $T(X)$.

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1 Introduction

Let $T(X)$ denote the semigroup of transformations from a set X into itself under composition of mappings. We call $T(X)$ the full transformation semigroup on X. Its subsemigroups of $T(X)$ have been widely investigated. For examples, Pei [1] has introduced a family of subsemigroups of $T(X)$ defined by

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}$$

where E is an arbitrary equivalence relation on X. When $(X, \leq)$ is a partially ordered set, Saitô et al. [2] have considered a family of subsemigroups of $T(X)$ as follows:

$$T_O(X) = \{\alpha \in T(X) : \forall x, y \in X, x \leq y \Rightarrow x\alpha \leq y\alpha\}.$$

In this paper the set X under consideration is a partially ordered set with E an arbitrary equivalence relation on X. We define a family of subsemigroups of $T(X)$ as follows:

$$T_{EO}(X) = \{\alpha \in T(X) : \forall\ x, y \in X, (x, y) \in E, x \leq y \Rightarrow (x\alpha, y\alpha) \in E, x\alpha \leq y\alpha\}.$$

Then $T_E(X) \cap T_O(X) \subseteq T_{EO}(X)$.

In this paper, we consider relationships between $T_{EO}(X)$, $T_E(X)$ and $T_O(X)$.

Following the usual terminology for a partially ordered set X, let us say that $a, b \in X$ are *comparable* if either $a \leq b$ or $b \leq a$, and *incomparable* if neither of these holds.

A family $\pi = \{A_i : i \in I\}$ of nonempty subsets of X is said to form a *partition of* X if $\cup\pi = X$ and for all $i, j \in I$, either $A_i = A_j$ or $A_i \cap A_j = \emptyset$.

2 Main Results

In this section, we investigate the condition under which subsemigroups of $T(X)$ are related.

Proposition 2.1. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. Then $T_{EO}(X) = T_O(X)$ if and only if $\bigcup \mathcal{K} \subseteq E$ where $\mathcal{K} = \{\mathcal{C} \times \mathcal{C} : \mathcal{C}$ is a subchain of $X\}$.*

Proof. Suppose that there exists $(a, b) \in \bigcup \mathcal{K}$ such that $(a, b) \notin E$. Then $a \in A$ and $b \in B$ for some $A, B \in X/E$. Since $(a, b) \in \bigcup \mathcal{K}$, $a, b \in \mathcal{C}$ for some subchain $\mathcal{C}$ of X. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} a & \text{if } x \in B; \\ b & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ be such that $x \leq y$ and $(x, y) \in E$. By definition of α, we deduce that

$$(x\alpha, y\alpha) = \begin{cases} (a, a) \in E & \text{if } x, y \in B; \\ (b, b) \in E & \text{otherwise.} \end{cases}$$

It follows that $\alpha \in T_{EO}(X)$. Since a and b are comparable, we may assume that $a < b$. Then we have $a\alpha = b \not\leq a = b\alpha$. Hence $\alpha \notin T_O(X)$.

Conversely, assume that $\bigcup \mathcal{K} \subseteq E$ where $\mathcal{K} = \{\mathcal{C} \times \mathcal{C} : \mathcal{C}$ is a subchain of $X\}$. To show that $T_{EO}(X) = T_O(X)$, let $\alpha \in T_{EO}(X)$ and $a, b \in X$ with $a \leq b$. Thus $a, b \in \mathcal{C}$ for some subchain $\mathcal{C}$ of X. By assumption, we have $(a, b) \in E$. Since $\alpha \in T_{EO}(X)$, $a\alpha \leq b\alpha$. Hence $\alpha \in T_O(X)$. Next, let $\alpha \in T_O(X)$ and $x, y \in X$ with $(x, y) \in E$ and $x \leq y$. Since $\alpha \in T_O(X)$, $x\alpha \leq y\alpha$ which implies that $x\alpha, y\alpha \in \mathcal{C}$ for some subchain $\mathcal{C}$ of X. It follows by assumption that $(x\alpha, y\alpha) \in E$. Hence $\alpha \in T_{EO}(X)$. $\square$

Proposition 2.2. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. Then $T_{EO}(X) = T(X)$ if and only if for every two distinct a, b in X, $(a, b) \in E$ implies that a and b are incomparable.*

Proof. Suppose that there exist distinct elements a, b in X such that $(a, b) \in E$ and a and b are comparable. We may assume that $a < b$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a & \text{if } x = b; \\ b & \text{otherwise.} \end{cases}$$

By definition of β, we then have $a\beta = b \not\leq a = b\beta$. This means that $\beta \notin T_{EO}(X)$.

Conversely, assume that for every two distinct a, b in X, $(a, b) \in E$ implies that a and b are incomparable. Let $\alpha \in T(X)$ and $x, y \in X$ with $(x, y) \in E$ and $x \leq y$. We deduce that $x = y$ which implies that $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$. Therefore $\alpha \in T_{EO}(X)$. □

Corollary 2.3. *Let X be a partially ordered set and E an arbitrary equivalence relation on X.*

(1) *If $E = X \times X$, then $T_{EO}(X) = T_O(X)$ and $T_E(X) = T(X)$.*

(2) *If $E = I_X$, then $T_E(X) = T_{EO}(X) = T(X)$.*

Theorem 2.4. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. If $T_{EO}(X) \subseteq T_E(X)$, then*

(1) *$E = X \times X$ or*

(2) *for each $A \in X/E$ and arbitrary partition $\{P, Q\}$ of A, there exist $x \in P, y \in Q$ such that x and y are comparable.*

Proof. Suppose that $E \neq X \times X$ and (2) is not true. Then there exists $A \in X/E$ and a partition $\{P, Q\}$ of A such that a and b are incomparable for all $a \in P, b \in Q$. Since $E \neq X \times X$, choose $B \in X/E$ such that $B \neq A$ and fix $b \in B$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b & \text{if } x \in P; \\ x & \text{otherwise.} \end{cases}$$

To show that $\alpha \in T_{EO}(X)$, let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. Hence $x, y \in D$ for some $D \in X/E$.

Case 1. $D \neq A$. Then $x, y \notin P$. By definition of α, $x\alpha = x$ and $y\alpha = y$. Hence $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$.

Case 2. $D = A$. Since $x \leq y$ and $\{P, Q\}$ is a partition of A, either $x, y \in P$ or $x, y \in Q$. This implies that $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$.

It follows by two cases that $\alpha \in T_{EO}(X)$. Notice that for any $x \in P$ and $y \in Q$, $(x, y) \in E$ but $(x\alpha, y\alpha) = (b, y) \notin E$. Therefore $\alpha \notin T_E(X)$. □

Theorem 2.5. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. Suppose that for every $A \in X/E$ and $x, y \in A$, there exist subchains $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n$ of A for some positive integer n such that $x \in \mathcal{C}_1, y \in \mathcal{C}_n$ and $\mathcal{C}_i \cap \mathcal{C}_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$. Then $T_{EO}(X) \subseteq T_E(X)$.*

Proof. Suppose that for every $A \in X/E$ and $x, y \in A$, there exist subchains $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n$ of A for some positive integer n such that $x \in \mathcal{C}_1, y \in \mathcal{C}_n$ and $\mathcal{C}_i \cap \mathcal{C}_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$. Let $\alpha \in T_{EO}(X)$ and $(x, y) \in E$. Hence $x, y \in A$ for some $A \in X/E$. It follows by assumption that there exist subchains $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n$ of A for some positive integer n such that $x \in \mathcal{C}_1, y \in \mathcal{C}_n$ and $\mathcal{C}_i \cap \mathcal{C}_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$. Choose $c_i \in \mathcal{C}_i \cap \mathcal{C}_{i+1}$ for all $i = 1, 2, \ldots, n-1$. Since $x, c_1 \in \mathcal{C}_1$, x and c_1 are comparable. Assume that $x \leq c_1$. By $\alpha \in T_{EO}(X)$, we deduce that $(x\alpha, c_1\alpha) \in E$. For each $i = 1, 2, \ldots, n-1$, we have $c_i, c_{i+1} \in \mathcal{C}_{i+1}$. We may assume that $c_i \leq c_{i+1}$. Since $(c_i, c_{i+1}) \in E$ and $\alpha \in T_{EO}(X)$, $(c_i\alpha, c_{i+1}\alpha) \in E$. Similarly, we have that $(c_n\alpha, y\alpha) \in E$. It follows by transitive of E that $(x\alpha, y\alpha) \in E$. This proves that $\alpha \in T_E(X)$. □

Example 1. *Let $X = \{a_1, a_2, a_3, b\}$ and $E = \{a_1, a_2\} \times \{a_1, a_2\} \cup \{a_3, b\} \times \{a_3, b\}$. We define $\leq = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (b, b)\}$. Then X is a partially ordered set and E is an equivalence relation on X. Define $\alpha, \beta, \delta \in T(X)$ by*

$$x\alpha = \begin{cases} a_3 & \text{if } x = a_1; \\ b & \text{if } x = a_2; \\ x & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} a_2 & \text{if } x = a_1; \\ a_3 & \text{if } x = a_2; \\ x & \text{otherwise} \end{cases}$$

and

$$x\delta = \begin{cases} a_1 & \text{if } x = a_3; \\ x & \text{otherwise.} \end{cases}$$

It is easy to verify that $\alpha \in T_E(X) \setminus (T_O(X) \cup T_{EO}(X))$, $\beta \in T_O(X) \setminus (T_E(X) \cup T_{EO}(X))$ and $\delta \in T_{EO}(X) \setminus (T_O(X) \cup T_E(X))$.

References

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