**International Mathematical Forum, Vol. 7, 2012, no. 45, 2239 - 2252**

# Some Properties of Operator Classes $(M,k)^*$, $A^*[k]$ and $k-$* Paranormal Operator

**Fahri Marevci**

Faculty of Education

University of Prishtina, Prishtinë 10000, Kosovë

fahri_marevci@hotmail.com

**Muhib Lohaj**

Department of Mathematics and Computer Sciences, Avenue Mother Teresa Nr. 5

University of Prishtina, Prishtinë 10000, Kosovë

muhib_lohaj@yahoo.com

**Abstract**

In this article we have proved that every operator in $(M,k)^*$ class for $k \geq 2$ is a $k$-* paranormal operator, also we give some properties about these classes. We showed that for every non-zero operators $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$ their tensor product $T_1 \otimes T_2$ belongs to the $A^*[k]$ class, if and only if, $T_1$ and $T_2$ belong to the $A^*[k]$ class, for $k \geq 1$.

**Mathematics Subject Classification:** 47B20, 47B37

**Keywords:** Hilbert space, classes $(M,k)^*$ , $A^*[k]$ and $k$ -*paranormal operators

## 1. Introduction

Let us denote by $H$ the complex Hilbert space and $B(H)$ the space of all bounded linear operators defined in Hilbert space $H$. In the following we will mention some known classes of operators defined in Hilbert space $H$. Let $T$ be an element in the algebra of bounded operators $B(H)$. The operator $T$ is called quasi-normal if $T(T^*T)=(T^*T)T$ , it is hyponormal if $T^*T \geq TT^*$ , which is equivalent to the condition $\|Tx\| \geq \|T^*x\|$ , for all $x$ in $H$. We say that an operator $T$ is quasi-hyponormal if the following condition: $T^{*2}T^2 \geq (T^*T)^2$ holds and the last one is equivalent with $\|T^2x\| \geq \|T^*Tx\|$, for all $x$ in $H$. We say that an operator $T$ is of $(M,k)$ class if $T^{*k}T^k \geq (T^*T)^k$, for $k \geq 2$, which is equivalent to the condition $\|T^kx\| \geq \left\|(T^*T)^{\frac{k}{2}}x\right\|$ , for all $x$ in $H$ and $k \geq 2$, (see [8]). It is known that the $(M,2)$ class coincides with the class of quasi-hyponormal operators. But, the class of hyponormal operators does not coincide with $(M,k)$, for any $k$, (see [7]). We say that an operator $T$ is of $(M,k)^*$ class if $T^{*k}T^k \geq (TT^*)^k$, for $k \geq 1$, which is equivalent to the condition $\|T^kx\| \geq \left\|(TT^*)^{\frac{k}{2}}x\right\|$ , for all $x$ in $H$ and $k \geq 1$, (see [8]). It is known that the $(M,1)^*$ class coincides with the class of hyponormal operators. The operator $T$

is called $k-*$ paranormal if it satisfies the following condition $\|T^k x\| \geq \|T^* x\|^k$, for all unit vectors $x$ in $H$ and $k \geq 2$. The operator $T \in B(H)$ is of $A^*[k]$ class if $|T^k|^{\frac{2}{k}} \geq |T^*|^2$, for $k \geq 1$, ($k$- an integer ). The spectrum, the point spectrum, the approximate point spectrum of an operator $T$ are denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, respectively.

***Theorem A. ( Hölder-McCarthy inequality [1]).*** Let $A$ be a positive operator. Then the following inequalities hold for all $x$ in $H$

*i)* $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \|x\|^{2(1-r)}$, for $0 < r \leq 1$,

*ii)* $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r \|x\|^{2(1-r)}$, for $r \geq 1$.

***Proposition A. [6].*** Let $T_1, T_2 \in B(H_1)$, $S_1, S_2 \in B(H_2)$ be non-negative operators. If $T_1$ and $S_1$ are non–zero operators, then the following assertions are equivalent

*i)* $T_1 \otimes S_1 \leq T_2 \otimes S_2$,

*ii)* There exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1} S_2$.

***Proposition B. [3]***. For every $\alpha, \beta \in C$, $T, T_1, T_2 \in B(H_1)$ and $S, S_1, S_2 \in B(H_2)$,

*i).* $\alpha\beta(T \otimes S) = \alpha T \otimes \beta S$,

*ii).* $(T_1 + T_2) \otimes (S_1 + S_2) = T_1 \otimes S_1 + T_2 \otimes S_1 + T_1 \otimes S_2 + T_2 \otimes S_2$,

*iv).* $(T_1 \otimes S_1)(T_2 \otimes S_2) = T_1 T_2 \otimes S_1 S_2$, *v).* $(T\otimes S)^* = T^* \otimes S^*$,

*vi).* $\|T\otimes S\| = \|T\|\,\|S\|$.

If $T$ and $S$ are invertible , then so is $T\otimes S$ and

*vii).* $(T\otimes S)^{-1} = T^{-1}\otimes S^{-1}$.

## 2. Classes of operators $(M,k)^*$, $A^*[k]$ and $k$-*paranormal in Hilbert space

In this section we will show some properties of $(M,k)^*$, $A^*[k]$ classes and $k$-*paranormal operators.

***Proposition 2.1.*** For each positive integer $k\ge 1$ an operator $T$ belongs to class $(M,k)^*$ if and only if

$$T^{*k}T^k + 2\lambda(TT^*)^k + \lambda^2 T^{*k}T^k \ge 0,$$

holds for all $\lambda\in R$.

***Proof.*** Let $\lambda\in R$ and $x\in H$ be given. Then $T\in(M,k)^*$, if and only if

$$\left\|(TT^*)^{\frac{k}{2}}x\right\| \le \left\|T^k x\right\| \Leftrightarrow 4\left\|(TT^*)^{\frac{k}{2}}x\right\|^4 - 4\cdot\left\|T^k x\right\|^2\cdot\left\|T^k x\right\|^2 \le 0$$

$$\Leftrightarrow \left\|T^k x\right\|^2 + 2\lambda\left\|(TT^*)^{\frac{k}{2}}x\right\|^2 + \lambda^2\left\|T^k x\right\|^2 \ge 0$$

$$\Leftrightarrow \langle(T^k x, T^k x\rangle + 2\lambda\langle(TT^*)^{\frac{k}{2}}x, (TT^*)^{\frac{k}{2}}x\rangle + \lambda^2\langle T^k x, T^k x\rangle \ge 0$$

$$\Leftrightarrow \langle(T^{*k}T^k x\rangle + 2\lambda\langle(TT^*)^k x, x\rangle + \lambda^2\langle T^{*^k}T^k x, x\rangle \ge 0$$

$$\Leftrightarrow \langle (T^{*k}T^k + 2\lambda(TT^*)^k + \lambda^2 T^{*k}T^k)x, x\rangle \geq 0$$

$$\Leftrightarrow T^{*k}T^k + 2\lambda(TT^*)^k + \lambda^2 T^{*k}T^k \geq 0,$$

by which the proof is completed.■

**Corollary 2.1.** If $k = 1$, we get the following relation $T^*T \geq TT^*$ if and only if $T^*T + 2\lambda TT^* + \lambda^2 T^*T \geq 0$, for all $\lambda \in R$, which is the definition of the hyponormal operator.

***Lemma 2.1.*** If $T$ is a bilateral weighted shift operator, with weighted sequence $\omega_n$, $(Te_n = \omega_n e_{n+1})$, then it is of $(M,k)^*$ class if and only if

$$|\omega_n| \cdot |\omega_{n+1}| \cdot \ldots \cdot |\omega_{n+k-1}| \geq |\omega_{n-1}|^k, \text{ for } n \in Z \textit{ and } k \geq 1.$$

***Proof.*** The proof follows immediately from the definition of $(M,k)^*$ class.■

***Example 2.1.*** Let $T \in B(H)$ be a bilateral weighted shift with weighted sequence $(\omega_n)$ given as follows

$$\omega_n = \begin{cases} \frac{1}{2}, & \textit{for } n \leq -1 \\ 2, & \textit{for } n = 0 \\ \frac{1}{2}, & \textit{for } n = 1 \\ 4, & \textit{for } n = 2 \\ 16, & \textit{for } n \geq 3. \end{cases}$$

After some calculations, we have that $T \in (M,3)^*$, but $T \notin (M,k)^*$, for $k = 1,2$ (see lemma 2.1.).

***Example 2.2.*** Let $T \in B(H)$ be a bilateral weighted shift with weighted sequence $(\omega_n)$ given by the formula

$$\omega_n = \begin{cases} \frac{1}{3}, \ for \ n \le -1 \\ 1, \ for \ n = 0 \\ \frac{1}{3}, \ for \ n = 1 \\ 3, \ for \ \ n = 2 \\ \frac{1}{9}, \ for \ n = 3 \\ 729, \ for \ n \ge 4. \end{cases}$$

After some calculations, it follows that $T \in (M,2)^*$, but $T \notin (M,k)^*$, for $k = 1,3$ (see lemma 2.1.).

***Example 2.3.*** **(See Theorem 2.3. in [10]).** Let $T \in B(H)$ be a bilateral weighted shift with weighted sequence $(\omega_n)$ given as follows

$$\omega_n = \begin{cases} \frac{1}{2}, \ for \ n \le -1 \\ \frac{1}{\sqrt{3}}, \ for \ n = 0 \\ \frac{n}{n+1}, \ for \ n \ge 1. \end{cases}$$

For $n = 1$, by Lemma 2.1, we have $\omega_1 \cdot \omega_2 \cdot ... \cdot \omega_k \ge \omega_0^k$, for $k \ge 1$. Therefore $T$ is in $(M,k)^*$ class, for $k \ge 2$ but it is not hyponormal, because $\omega_0 > \omega_1$.

***Proposition 2.2.*** For each positive $k \ge 2$, $T$ is a $k-$*-paranormal if and only if

$$T^{*k}T^k - k\lambda^{k-1}TT^* + (k-1)\lambda^k I \ge 0, \quad (4)$$

holds for all $\lambda > 0$.

***Proof.*** Suppose $T$ is a $k-$*paranormal and $x$ is a unit vector in $H$. By generalized arithmetic-geometric mean inequality, we have

$$\frac{1}{k}\langle\lambda^{1-k}T^{k}x,T^{k}x\rangle+\frac{k-1}{k}\langle\lambda x,x\rangle\geq\langle\lambda^{1-k}T^{k}x,T^{k}x\rangle^{\frac{1}{k}}\langle\lambda x,x\rangle^{\frac{k-1}{k}}$$

$$=\lambda^{\frac{1-k}{k}}\langle T^{k}x,T^{k}x\rangle^{\frac{1}{k}}\lambda^{\frac{k-1}{k}}\langle x,x\rangle^{\frac{k-1}{k}}$$

$$=\left\|T^{k}x\right\|^{\frac{2}{k}}\geq\left\|T^{*}x\right\|^{2}=\langle TT^{*}x,x\rangle .$$

Hence

$$\lambda^{1-k}\langle T^{*k}T^{k}x,x\rangle+\lambda(k-1)\langle x,x\rangle-k\langle TT^{*}x,x\rangle\geq 0$$

$$\langle(T^{*k}T^{k}-k\lambda^{k-1}TT^{*}+(k-1)\lambda^{k}I)x,x\rangle\geq 0 .$$

Thus $T^{*k}T^{k}-k\lambda^{k-1}TT^{*}+(k-1)\lambda^{k}I\geq 0$, for all $\lambda>0$.

*Conversely.* Let $x\in H$, $\|x\|=1$ and $\lambda=\langle T^{*}x,T^{*}x\rangle>0$. Then if we put $\lambda=\langle T^{*}x,T^{*}x\rangle>0$ in (4) we have

$$\langle T^{k}x,T^{k}x\rangle-k\langle T^{*}x,T^{*}x\rangle^{k-1}\langle T^{*}x,T^{*}x\rangle+(k-1)\langle T^{*}x,T^{*}x\rangle^{k}\geq 0$$

$$\left\|T^{k}x\right\|^{2}-k\left\|T^{*}x\right\|^{2k}+(k-1)\left\|T^{*}x\right\|^{2k}\geq 0$$

$$\left\|T^{k}x\right\|^{2}-\left\|T^{*}x\right\|^{2k}\geq 0 .$$

Therefore $\left\|T^{k}x\right\|\geq\left\|T^{*}x\right\|^{k}$, respectively $T$ is a $k-$*paranormal. ■

**Corollary 2.2.** If $k=2$, we get the following relation $\left\|T^{*}x\right\|^{2}\leq\left\|T^{2}x\right\|$ if and only if $T^{*2}T^{2}-2\lambda TT^{*}+\lambda^{2}I\geq 0$, for all $\lambda>0$ and $x\in H, \|x\|=1$, which is the definition of the *paranormal operator.

***Proposition 2.3.*** Let $T$ be a regular $k-$*paranormal operator. Then the approximate point spectrum lies in the disc

$$\sigma_{ap}(T) \subseteq \left\{ \lambda \in C : \frac{\|T\|}{\left(\|T^{*-1}\| \cdot \|T\|\right)^{k}} \leq |\lambda| \leq \|T\| \right\}.$$

***Proof.*** Suppose $T$ is a regular $k-$*paranormal operator, for $k \geq 2$. Then for every unit vector $x$ in $H$, we have

$$\|x\|^{k} = \|T^{*-1}T^{*}x\|^{k} \leq \|T^{*-1}\|^{k} \cdot \|T^{*}x\|^{k} \leq \|T^{*-1}\|^{k} \cdot \|T^{k}x\|$$

$$1 \leq \|T^{*-1}\|^{k} \cdot \|T^{k-1}\| \cdot \|Tx\|$$

$$\|Tx\| \geq \frac{1}{\|T^{*-1}\|^{k}\|T^{k-1}\|} \geq \frac{1}{\|T^{*-1}\|^{k}\|T\|^{k-1}}. \qquad (5)$$

Now, assume that $\lambda \in \sigma_{ap}$. Then there exists a sequence $(x_n)$, $\|x_n\| = 1$, such that $\|(T-\lambda)x_n\| \to 0$, when $n \to \infty$. Therefore by (5) we have

$$\|Tx_n - \lambda x_n\| \geq \|Tx_n\| - |\lambda|\|x_n\| \geq \frac{1}{\|T^{*-1}\|^{k}\|T\|^{k-1}} - |\lambda|. \qquad (6)$$

Now, when $n \to \infty$, from relation (6) we have

$$|\lambda| \geq \frac{1}{\|T^{*-1}\|^{k}\|T\|^{k-1}} = \frac{\|T\|}{\left(\|T^{*-1}\| \cdot \|T\|\right)^{k}},$$

respectively

$$\sigma_{ap}(T) \subseteq \left\{ \lambda \in C : \frac{\|T\|}{\left(\|T^{*-1}\| \cdot \|T\|\right)^{k}} \leq |\lambda| \leq \|T\| \right\}.$$

Therefore the proof is completed. ■

**Corollary 2.3.** Let $T$ be a regular * paranormal operator. Then the following relation

$$\sigma_{ap}(T) \subseteq \left\{ \lambda \in C : \frac{\|T\|}{\left(\|T^{*-1}\| \cdot \|T\|\right)^2} \leq |\lambda| \leq \|T\| \right\}$$

holds.

***Proposition 2.4.*** Let $T$ be a regular $k-$paranormal operator. Then the approximate point spectrum lies in the disc

$$\sigma_{ap}(T) \subseteq \left\{ \lambda \in C : \frac{\|T\|}{\left(\|T^{-1}\| \cdot \|T\|\right)^k} \leq |\lambda| \leq \|T\| \right\}.$$

***Proof.*** The proof is similar with the proof of the Proposition 2.3. ■

***Theorem 2.1.*** If $T \in (M,k)^*$, $k \geq 2$, then $T$ is $k-$*paranormal operator.

***Proof.*** Firstly we will prove that every operator $T$ that belongs to the $(M,k)^*$ it belongs to $A^*[k]$ class, also. If $T \in (M,k)^*$ then we have

$$\left|T^k\right|^{\frac{2}{k}} = \left(T^{*k}T^k\right)^{\frac{1}{k}} \geq \left(TT^*\right)^{k \cdot \frac{1}{k}} = TT^* = \left|T^*\right|^2.$$

Hence $\left|T^k\right|^{\frac{2}{k}} \geq \left|T^*\right|^2$, for $k \geq 2$, respectively $T \in A^*[k]$.

In the following let $T \in A^*[k]$, $k \geq 2$. Then for every unit vector $x \in H$ we have

$$\left\|T^k x\right\|^2 = \langle T^k x, T^k x\rangle = \langle T^{*k}T^k x, x\rangle = \langle \left|T^k\right|^2 x, x\rangle = \langle \left|T^k\right|^{\frac{2}{k} \cdot k} x, x\rangle$$

$$\geq \langle \left|T^k\right|^{\frac{2}{k}} x, x\rangle^k \quad \textbf{(}\text{Hölder-McCarthy inequality}\textbf{)}$$

$$\geq \langle \left|T^*\right|^2 x, x\rangle^k = \langle TT^* x, x\rangle^k = \langle T^* x, T^* x\rangle^k = \left\|T^* x\right\|^{2k}.$$

Finally, $\left\|T^k x\right\| \geq \left\|T^* x\right\|^k$, for $k \geq 2$, respectively $T$ is $k-$*paranormal, and the proof is completed. ■

***Corollary 2.4.*** Let $T \in B(H)$ be a quasi-normal operator. If $T$ is a hyponormal operator then $T$ is a $k$-* paranormal.

***Proof.*** The proof of this corollary is the direct consequence of the theorem 3.11 in $[7]$ and theorem 2.1. ■

***Corollary 2.5.*** If $T \in (M, k+1)$ has dense range in $H$, then it is $k$-*paranormal.

***Proof.*** This proof follows from theorem 3.8 in $[7]$ and theorem 2.1. ■

***Remark 2.1.*** For $k = 2$ we obtain theorem 3.7 in $[7]$.

**Theorem 2.2**. Let $T$ belongs to the $A^*[k]$ class, where $k \geq 1$ and let $\alpha$ be an eigenvalue of the operator $T$. Further, let $T = \begin{bmatrix} \alpha I & 0 \\ 0 & T_1 \end{bmatrix}$ be the matrix representation in $H = \ker(T - \alpha I) \oplus \ker(T - \alpha I)^{\perp}$, were $T_1$ is defined in the subspace $\ker(T - \alpha I)^{\perp}$. Then $T_1 \in A^*[k]$, for $k \geq 1$ .

***Proof*.** Assume that $T \in A^*[k]$, $k \geq 1$, then we have

$$0 \leq \left|T^k\right|^{\frac{2}{k}} - \left|T^*\right|^2 = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & \left|T_1^k\right|^{\frac{2}{k}} \end{bmatrix} - \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & \left|T_1^*\right|^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \left|T_1^k\right|^{\frac{2}{k}} - \left|T_1^*\right|^2 \end{bmatrix}.$$

Since $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \geq 0$ if end only if $A \geq 0$ and $C \geq 0$, it follows that

$$\begin{bmatrix} 0 & 0 \\ 0 & \left|T_1^k\right|^{\frac{2}{k}} - \left|T_1^*\right|^2 \end{bmatrix} \geq 0 \Leftrightarrow \left|T_1^k\right|^{\frac{2}{k}} - \left|T_1^*\right|^2 \geq 0,$$

Hence $\left|T_1^k\right|^{\frac{2}{k}} \geq \left|T_1^*\right|^2$, for $k \geq 1$, respectively $T_1 \in A^*[k]$ .■

Let $H_1$ and $H_2$ denotes the Hilbert spaces. For given non-zero operators $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$, $T_1 \otimes T_2$ denotes the tensor on the product space $H_1 \otimes H_2$. Tensor product of two non-zero operators satisfy the following equalities

a) $(T_1 \otimes T_2)^*(T_1 \otimes T_2) = T_1^*T_1 \otimes T_2^*T_2$

b). $|T_1 \otimes T_2|^p = |T_1|^p \otimes |T_2|^p$ for any positive real number $p$.

**Theorem 2.3.** Let $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$ are non-zero operators. Then $T_1 \otimes T_2$ belongs to the $A^*[k]$ class, if and only if, $T_1$ and $T_2$ belong to the $A^*[k]$ class, for $k \geq 1$.

***Proof.*** Let we suppose that $T_1$ and $T_2$ belong to the $A^*[k]$ class for $k \geq 1$. Then by Proposition B and from the properties of the tensor product of operators we have

$$\left|(T_1 \otimes T_2)^k\right|^{\frac{2}{k}} = \left|T_1^k \otimes T_2^k\right|^{\frac{2}{k}} = \left|T_1^k\right|^{\frac{2}{k}} \otimes \left|T_2^k\right|^{\frac{2}{k}}$$

$$\geq \left|T_1^*\right|^2 \otimes \left|T_2^*\right|^2 = \left|T_1^* \otimes T_2^*\right|^2 = \left|(T_1 \otimes T_2)^*\right|^2.$$

Thus $\left|(T_1 \otimes T_2)^k\right|^{\frac{2}{k}} \geq \left|(T_1 \otimes T_2)^*\right|^2$, respectively $T_1 \otimes T_2$ belongs to the $A^*[k]$ class.

*Conversely*. Assume that $T_1 \otimes T_2$ belongs to the $A^*[k]$ class, then by

Proposition B it follows that

$$|T_1^*|^2 \otimes |T_2^*|^2 = |T_1^* \otimes T_2^*|^2 = |(T_1 \otimes T_2)^*|^2 \le |(T_1 \otimes T_2)^k|^{\frac{2}{k}} = |T_1^k|^{\frac{2}{k}} \otimes |T_2^k|^{\frac{2}{k}}.$$

Now by Proposition A , there exists a positive real number $c$ such that

$$|T_1^*|^2 \le c|T_1^k|^{\frac{2}{k}} \text{ and } |T_2^*|^2 \le c^{-1}|T_2^k|^{\frac{2}{k}}. \tag{7}$$

Hence , by Theorem A for every unit vector $x \in H_1$, we have

$$\begin{aligned}\|T_1^*\|^2 &= \sup_{\|x\|=1}\langle T_1^* x, T_1^* x\rangle = \sup_{\|x\|=1}\langle T_1 T_1^* x, x\rangle \\ &= \sup_{\|x\|=1}\langle |T_1^*|^2 x, x\rangle \le \sup_{\|x\|=1}\langle c|T_1^k|^{\frac{2}{k}} x, x\rangle \\ &= c\sup_{\|x\|=1}\langle |T_1^k|^{\frac{2}{k}} x, x\rangle \le c\sup_{\|x\|=1}\langle |T_1^k|^2 x, x\rangle^{\frac{1}{k}} \\ &= c\sup_{\|x\|=1}\langle T_1^k x, T_1^k x\rangle^{\frac{1}{k}} = c\|T_1^k\|^{\frac{2}{k}} \le c\|T_1\|^2 = c\|T_1^*\|^2.\end{aligned}$$

Hence, $\|T_1^*\| \le \sqrt{c}\,\|T_1^*\|$ in $H_1$ . Similarly we prove that $\|T_2^*\| \le \sqrt{c^{-1}}\,\|T_2^*\|$ in $H_2$. Finaly, from the above two inequalities and from inequalities (7), it follows that $c = 1$ and therefore $T_1, T_2 \in A^*[k]$ . ■

## REFERENCES

[1] C. A. McCarthy, $c_p$, Israel J. Math., 5 (1967), 249-271.

[2] C. S. Kubrusly and N. Levan, *Preservation of tensor sum and tensor product*, Acta Math. Univ. Comenianae, Vol. LXXX, 1 (2011), pp. 133-142.

[3] Carlos S. Kubrusly, *A concise introduction to tensor product*, Far East J. Math. Sci. 22(2006), 137-174.

[4]. In Hyoun Kim*, *On sprctral continuities and tensor products of operators*, Journal of the Chungcheong Mathematical Society, Volum 24, No. 1, March 2011.

[5] Jiangtao Yuan and Zongsheng Gao , *Weyls Spectrum of Class* $A(n)$ *and* $n-$*Paranormal Operators* , Interg. Equ. oper. Theory 60 (2008), 289-298.

[6] J. Stochel, *Seminormality of operators from their tensor product*. Proc. Amer. Math. Soc. 124 (1996), 435-440.

[7] N. Chennappan, S. Karthikeyan, * *Paranormal composition operators*, Indian J. Pure Appl. Math., 31 (2000), no. 6, 591-601.

[8] N. L. Braha, M. Lohaj, F. H. Marevci, Sh. Lohaj, *Some properties of paranormal and hyponormal operators*, Bull. Math. Anal. Appl., v.1, Issue 2, (2009), 23-35.

[9] P.R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, 1967.

[10]. Pushpa R, Suri and N. Singh, *Some results on* $k$*-quasi- -hyponormal operators*, Bull. Austral. Math. Soc. Vol. 32 (1985), 351-355.

[11] S. C. Arora and Ramesh Kumar, $M$*-Paranormal operators*, Publ. Inst. Math., Nouvelle serie 29 (43) (1981), 5-13.

[12] S. C. Arora and J. K. Thukral, $M$ - * *Paranormal operators*, Glas. Math. Ser. III, Vol. 22, No.1. (1987), 123-129.

[13]. S. Panayappan, N. Jayanthi and D. Sumathi, *Weyl's Theorem and Tensor Product for Class* $A_k$ *Operators,* Pure Mathematical Sciences, Vol. 1, 2012, no. 1, 13-23.

[14] Youngsik Park and Cheonseoung Ryoo, *Some results on* $k^*$ *– paranormal operators*, East Asian Math J. 14 (1998), No 1, pp. 27- 34.

**Received: April, 2012**