**International Mathematical Forum, Vol. 7, 2012, no. 45, 2205 - 2209**

# A (Very) Simple Proof that $H^1(G,V) = (0)$ for a Compact, or Connected Semi-Simple Group

**Ioannis Farmakis**

Brooklyn College
The City University of New York (CUNY)
2900 Bedford Ave., Brooklyn, NY 11210, USA
IFarmakis@brooklyn.cuny.edu

**Abstract**

In this note we give a (very) simple proof of the known fact that the first cohomology group with coefficients in a finite dimensional real vector space $V$ of a compact, or of a connected semi-simple group $G$ must vanish.

**Mathematics Subject Classification:** 22C05, 22D12, 22E41, 22E46, 57T10

**Keywords:** Continuous group cohomology and its vanishing, continuous representation, affine action, fixed point, 1-cochain, 1-coboundary, compact group, semi-simple group

## 1 Introduction.

Here we prove that $H^1(G,V) = (0)$ for any compact, or connected semi-simple group $G$ with coefficients in a finite dimensional vector space $V$. The compact case is very well known, see for example Moskowitz, [7] pg. 334, and actually it is proved that for a compact group all the higher order cohomology groups vanish, even when $V$ is a Banach space (see [4], Theorem 6.0.3). Things are quite different when $G$ is a non compact connected semi-simple Lie group.

Here there only seem to be proofs of the following two results:

Let $G$ be a real, connected, semi-simple Lie group acting continuously on a Banach space $V$.

1. If none of the simple components is locally isomorphic to $SO_o(n,1)$ or $SU(n,1)$, then $H^1(G,V) = (0)$. (Erven-Kazdan [3] Chapter V).

2. If $G$ is simply connected, then $H^1(G,V) = (0)$ (S. Komy [5]).

For a counter example in the case of $SO_o(n,1)$ (which works equally well for $SU(n,1)$) see [4] pg. 118, or the original proof in [2].

For the reader convenience we recall the definition of the first cohomology group $H^1(G,V)$.

Let $G$ be a locally compact, second countable group and $\rho$ be a continuous representation of $G$ on a real finite dimensional vector space $V$. We we will use without distinction the notations $\rho(g)(v)$, or $g.v$ ($g \in G$ and $v \in V$). Then, the first cohomology group $H^1(G,V)$ is defined as follows:

**Definition 1.** *$H^1(G,V)$ is defined to be the quotient group $\mathcal{Z}^1/\mathcal{B}^1$, where $\mathcal{Z}^1$ is the space of the crossed homomorphisms (or 1-cocycles)*

$$\varphi : G \longrightarrow V \ : \ \varphi(gh) = \varphi(g) + g\varphi(h),$$

*and $\mathcal{B}^1$ consists of those $\varphi$ (or 1-coboundaries) of the form $\varphi(g) = g.v_0 - v_0$, for some $v_0$ in $V$ and all $g$ in $G$.*

Based in a geometric observation of Milnor ([6]), we shall give a very simple proof that $H^1(G,V) = (0)$ ($V = \mathbb{R}^n$) dealing with the compact and semi-simple cases simultaneously.

## 2 Main Theorem.

In fact, we have the following unifying result:

**Theorem 1.** *Let $G$ be a group all of whose finite dimensional real representations are completely reducible. Then for every finite dimensional representation of $G$ on $V$, $H^1(G,V) = (0)$.*

In particular,

**Corollary 1.** *If* $G$ *contains a connected semi-simple subgroup* $H$ *with* $G/H$ *either compact or of finite volume, then all finite dimensional real representations* $\rho$ *are completely reducible. (Of course if* $G$ *is compact, or connected semi-simple this is so. Hence, in all these cases* $H^1(G,V) = (0)$*).*

*Proof.* Since $H$ is connected semi-simple any continuous representation is completely reducible by H. Weyl's theorem (see e.g. [1] p. 175). Moreover, as is proved in Moskowitz [8] (Theorem 1, or Corollary 2 respectively), since $G/H$ is either compact, or of finite volume, $\rho$ must be completely reducible. □

To prove Theorem 1 we need the following:

**Definition 2.** *By an* invertible affine transformation *of a vector space* $V = \mathbb{R}^n$ *we mean a map* $V \longrightarrow V$ *given by* $x \mapsto Ax + b$*, where* $x, b \in V$*,* $A \in GL(V)$*.*

The next lemma is a slight modification of a result of Milnor (see [6] pg. 183).

**Lemma 1.** *If* $\rho$ *is completely reducible continuous representation of* $G$ *by affine transformations of* $V$*, then* $\rho$ *admits a fixed point.*

*Proof.* Identify the space $V$ with the hyperplane $\mathbb{R}^n \times \{1\}$ in $\mathbb{R}^{n+1}$. Now, any representation of $G$ by affine transformations of $V \times \{1\}$ extends uniquely to a linear representation of $G$ on $\mathbb{R}^{n+1}$. Indeed the map $x \mapsto Ax + b$, $x \in V$ extends to the map

$$\begin{pmatrix} x \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax+b \\ 1 \end{pmatrix}$$

which is linear. Since the linear subspace $\mathbb{R}^n \times \{1\}$ is invariant, by hypothesis, there exists a complementary $G$-invariant subspace $W$. Then, the intersection

$$W \cap (\mathbb{R}^n \times \{1\})$$

is a fixed point which is not the point (0) since (0) is not in this hyperplane. □

Turning to the proof of our theorem,

*Proof.* Let $\rho : G \longrightarrow GL(V)$ be a continuous linear representation of $G$ and $\varphi$ be a 1-cocycle. Define the affine map,

$$\rho_\varphi : G \longrightarrow \mathrm{Aff}(V) := G \ltimes GL(V),$$

given by

$$\rho_\varphi(g) : V \longrightarrow V \quad \text{such that} \quad \rho_\varphi(g)(v) := \rho(g)(v) + \varphi(g).$$

From the cocycle identity this map is a homomorphism. But by the Lemma 1, the affine map $\rho_\varphi$ has a fixed point. That is, there is a $v_0$ in $V$ with $\rho_\varphi(g)(v_0) = v_0$, for each $g \in G$. Then $\rho(g)(v_0) + \varphi(g) = v_0$ so that $\varphi$ is a 1-coboundary and $H^1(G, V) = (0)$. □

Brooklyn College, CUNY

## References


[1] H. Abbaspour, M. Moskowitz, *Basic Lie Theory*, World Scientific Publishing Co., 2007.

[2] P. Delorme, *1-cohomologie des representations unitaires des Groupes de Lie semi-simple et resolubles*, Bull. Soc. Math. France, vol. **105**, 1977, pp. 281-336.

[3] J. Erven, B.-J. Falkowski, *Low Order Cohomology and Applications*, Lecture Notes in mathematics, vol. **877**, Springer-Verlag, Berlin, 1980.

[4] I. Farmakis, *Cohomological Aspects of Complete Reducibility of Representations*, LAP LAMBERT Academic Publishing, Saarbrucken, 2010.

[5] S.R. Komy *On the first cohomology group for simply connected Lie groups*, J. Phys. A: Math. Gen. **18**, Issue 8, 1985, pp. 1159-1165.

[6] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Advances in Math., vol. **25**, 1977, pp. 178-187.

[7] M. Moskowitz, *Some remarks on Automorphisms of Bounded Displacement and Bounded Cocycles.*, Mh. Math., vol. **85**, pp. 323-336, 1978.

[8] M. Moskowitz, *Complete reducibility and Zariski density in linear Lie groups*, Mathematische Zeitschrift, vol. **232**, 1999, pp. 357-365.

**Received: April, 2012**