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The Graphs G that both G and $L(G)$ Contain 1-Perfect Codes

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Abstract

It is shown that both the graph G and its line graph $L(G)$ contain 1-perfect codes if and only if G has special structures which are mentioned in this paper.

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1 Introduction

In [2], N. Biggs introduced perfect codes in graphs as a generalization of the classical notions of perfect codes in the vector spaces [1, 14] and derived a necessary condition for the existence of perfect codes in distance-transitive graph. Unlike the case of distance-regular graphs where perfect codes are rare [3, 6, 15, 17], I. Dvořáková-Rulićová has constructed a $(d+1)$-regular graph containing a t-perfect code for any graph with the maximum degree d, where t is a given positive integer [4, Theorem 1]. J. Kratochvíl continued the study of perfect codes in graphs [9, 10] and together with his coleagues, he has shown that typical graphs don't contain 1-perfect codes [12] and the problem of perfect code recognition in graphs is NP-complete even when we are restricted to the regular graphs [11]. Recently, some authors have examined perfect codes in the Sierpińsky graphs and the direct product of the cycles [7, 8, 13, 16].

In this paper, we drive a necessary and sufficient condition for determining the graphs G that both G and the line graph $L(G)$ contain 1-perfect codes. It is shown that the such graph G must be one of the typical graphs which we define here.

2 Preliminaries

Let $G = (V(G), E(G))$ be an undirected graph with the vertex set $V(G)$ and edge set $E(G)$. The valency or degree of a vertex v is the number of edges containing v and the graph G is called regular if all the vertices of G have the same valency. The complete graph K_n is a graph on n vertices such that any two distinct vertices are adjacent. The line graph of G is the graph $L(G) = (E(G), V(G))$ where edges of G are vertices of $L(G)$ and two edges e and f of G are adjacent if they have a common vertex in G. A subgraph of G is a graph whose vertex set is a subset of $V(G)$ and whose adjacency relation is a subset of that of G restricted to this subset. A subgraph H of a graph G is said to be induced if, for any pair of vertices u and v of H, $uv \in E(H)$ if and only if $uv \in E(G)$.

Let t be a positive integer. A subset $C \subseteq V(G)$ with $|C| > 1$ is called a (non-trivial) t-perfect code if for each vertex $v \in V(G)$, there exists exactly one $c \in C$ such that $d(v, c) \leq t$. Elements of C are called code-vertices. Trivially, G contains a t-perfect code if and only if each of its connected components contains such sets and hence, we can only consider the connected graphs. It is easy to see for any graph G, a subset $C \subseteq V(G)$ with $|C| > 1$ and an integer $t > 0$, the following statements are equivalent:

(i) $C \subseteq V(G)$ is a t-perfect code in G.
(ii) $V(G) = \dot{\cup}_{c\in C} S_t(c)$, where $S_t(c) = \{v \in V(G)|d(v,c) \leq t\}$.
(iii) For any $v \in V(G)$, there exists $c \in C$ such that $d(v,c) \leq t$ and moreover, $d(c_1, c_2) \geq 2t+1$ for any two distinct code-vertices c_1 and c_2.

We know that 1-perfect codes are in fact the efficient domination sets in graphs [5] and every graph G is an induced subgraph of a graph G' containing a 1-perfect code. It is sufficient to take $G' = (V(G)\dot{\cup}V_1, E(G) \cup E_1)$, where $V_1 = \{v'|v \in V(G)\}$ and $E_1 = \{\{v, v'\}|v \in V(G)\}$, then V_1 is a 1-perfect code in G'.

3 Main result

At first, we need to bring some new definitions:

Definition 1 *A graph G is said to be of type (I), if there exist distinct vertices $c, u \in V(G)$ such that $E(G) = \{cv|v \in V(G) \setminus \{c\}\} \cup F$, where $F \subseteq \{uv|v \in V(G)\setminus\{c,u\}\}$, i.e., G can be obtained from a star graph by joining at most one 1-valency vertex to some other 1-valency vertices. In such graph, the vertices c and u are denoted by c_G and u_G, respectively.*

Definition 2 *A graph G is said to be of type (II), if there exists a vertex $c \in V(G)$ and a set W of some pairwise disjoint 2-subsets of $V(G) \setminus \{c\}$ such that $E(G) = \{cv|v \in V(G) \setminus \{c\}\} \cup E$, where $E = \{uv|\{u,v\} \in W\}$, i.e., G can be obtained from a star graph by joining some pairwise disjoint 1-valency vertices to each other. We denote the vertex c and the set W by c_G and $W(G)$, respectively.*

Definition 3 *A connected graph G is called a combination of the graphs $H_1, \ldots, H_m$ of type (I) and the graphs $K_1, \ldots, K_n$ of type (II), if*

$$V(G) = \dot{\bigcup_{1\leq i\leq m}} V(H_i) \dot{\cup} \dot{\bigcup_{1\leq i\leq n}} V(K_i)$$

and $E(G) = \bigcup_{1\leq i\leq m} E(H_i) \cup \bigcup_{1\leq i\leq n} E(K_i) \cup E_1 \cup E_2 \cup E_3$, where

$$E_1 \subseteq \bigcup_{1\leq i\neq j\leq m} \{u_{H_i}v|v \in V(H_j) \setminus \{c_{H_j}, u_{H_j}\}\},$$

E_2 connects all of the vertices of $\bigcup_{1\leq i\leq n} V(K_i) \setminus \cup W(K_i)$ pairwise disjoint to each other and

$$E_3 \subseteq \bigcup_{1\leq i\leq m, 1\leq j\leq n} \{uv|u \in V(H_i) \setminus \{c_{H_i}, u_{H_i}\}, v \in V(H_j) \setminus \{c_{H_j}\}\}.$$

Now, we can prove our main result:

Theorem 1 *Let G be a connected graph containing a 1-perfect code. Then the graph $L(G)$ contains a 1-perfect code if and only if G is a combination of the graphs of types (I) and (II).*

proof If the graph G contains a 1-perfect code $C = \{c_1, \ldots, c_n\}$, then we can write

$$V(G) = C \dot{\cup} \dot{\bigcup_{1\leq i\leq n}} \{v_{i,1}, \ldots, v_{i,n_i}\},$$

where $S_t(c_i) = \{c_i, v_{i,1}, \ldots, v_{i,n_i}\}$, and

$$E(G) = \bigcup_{1\leq i\leq n} \{c_i v_{i,1}, \ldots, c_i v_{i,n_i}\} \cup \bigcup_{1\leq i\leq n} E_i \cup F, \tag{1}$$

where $E_i \subseteq \{v_{i,k}v_{i,l} | 1 \leq k \neq l \leq n_i\}$ and

$$F \subseteq \{v_{i,k}v_{j,l} | 1 \leq i \neq j \leq n, 1 \leq k \leq n_i, 1 \leq l \leq n_j\}$$

(See Figure 1). Hence, by reindexing the vertices of line graph $L(G) = (\overline{V}, \overline{E})$, we have

$$\overline{V} = \dot{\bigcup_{1\leq i\leq n}} \{v'_{i,1}, \ldots, v'_{i,n_i}\} \dot{\cup} \dot{\bigcup_{1\leq i\leq n}} \{v''_{i,1}, \ldots, v''_{i,m_i}\} \dot{\cup} \{v'''_1, \ldots, v'''_l\},$$

where $n_i \geq 1$ and $m_i, l \geq 0$, and

$$\overline{E} = \bigcup_{1\leq i\leq n} E(K^{i,n_i}) \cup \bigcup_{1\leq i\leq n} \overline{E}_i \cup \overline{F},$$

where K^{i,n_i} is the complete graph on the vertices $\{v'_{i,1}, \ldots, v'_{i,n_i}\}$, $\overline{E}_i$ is the edge set of L_i which is the induced subgraph of $L(G)$ on $V_i := \{v'_{i,1}, \ldots, v'_{i,n_i}\} \cup \{v''_{i,1}, \ldots, v''_{i,m_i}\}$ such that

- for any $1 \leq j \leq m_i$, $v''_{i,j}$ is exactly connected to two vertices of $\{v'_{i,1}, \ldots, v'_{i,n_i}\}$,
- for any $1 \leq j \neq k \leq m_i$, $v''_{i,j}$ and $v''_{i,k}$ are adjacent if they are connected to a common vertex of $\{v'_{i,1}, \ldots, v'_{i,n_i}\}$,

and $\overline{F}$ is the edge set of an induced subgraph on $\overline{V}$ such that

- for any $1 \leq j \leq l$, v'''_j is exactly connected to two vertices $v'_{k_1,r}$ and $v'_{k_2,s}$ belonging to $V' := \bigcup_{1\leq i\leq n}\{v'_{i,1}, \ldots, v'_{i,n_i}\}$ with $k_1 \neq k_2$,
- for any $1 \leq j_1 \neq j_2 \leq l$, $1 \leq k \leq n$ and $1 \leq r \leq n_k$, the vertices v'''_{j_1} and v'''_{j_2}, and also v'''_{j_1} and $v''_{k,r}$ are connected if they are adjacent to a common vertex of V' (See Figure 2).

Now, suppose that the graph $L(G)$ contains a 1-perfect code D. Then it is easy to see that:

- if $v'_{i,j} \in D$, then $V_i = \{v'_{i,1}, \ldots, v'_{i,n_i}\} \cup \{v''_{i,k} | v'_{i,j}v''_{i,k} \in \overline{E}\}$ and $V_i \subseteq S_1(v'_{i,j})$.

• if $v''_{i,j} \in D$, then $D \cap V_i \subseteq \{v''_{i,1}, \ldots, v''_{i,m_i}\}$ and every two vertices of $D \cap V_i$ are not connected to a common vertex of $\{v'_{i,1}, \ldots, v'_{i,n_i}\}$.

Therefore, without loss of generality, there exists $0 \leq h \leq n$ such that

• $\overline{E}_i = \{v'_{i,1}v''_{i,1}, \ldots, v'_{i,1}v''_{i,m_i}\} \cup \{v''_{i,j}v''_{i,k} | 1 \leq j \neq k \leq m_i\}$ for any $1 \leq i \leq h$,

• $\overline{E}_i = \{v'_{i,2j-1}v''_{i,j}, v'_{i,2j}v''_{i,j} | 1 \leq j \leq m_i\}$ for any $h+1 \leq i \leq n$,

and there exist the integers $0 \leq l_1 \leq l_2 \leq l$ such that for every $1 \leq j \leq l$, v'''_j is exactly adjacent to two vertices $v'_{k_1,r}$ and $v'_{k_2,s}$ of the set V' with:

• if $1 \leq j \leq l_1$, then $1 \leq k_1 \neq k_2 \leq h$, $r = 1$ and $s \neq 1$.

• if $l_1 + 1 \leq j \leq l_2$, then $h+1 \leq k_1 \neq k_2 \leq n$, $2m_{k_1} + 1 \leq r \leq n_{k_1}$ and $2m_{k_2} + 1 \leq s \leq n_{k_2}$.

• if $l_2 + 1 \leq j \leq l$, then $1 \leq k_1 \leq h$, $h+1 \leq k_2 \leq n$, $2 \leq r \leq n_{k_1}$ and $1 \leq s \leq n_{k_2}$.

It is noticeable that every vertex $v'_{i,r}$ is connected to only one vertex of the set $\{v'''_{l_1+1}, \ldots, v'''_{l_2}\}$, where $h+1 \leq i \leq n$ and $2m_i + 1 \leq r \leq n_i$. Moreover, the perfect code D in $L(G)$ is

$$\{v'_{1,1}, \ldots, v'_{h,1}\} \cup \bigcup_{h+1 \leq i \leq n} \{v''_{i,1}, \ldots, v''_{i,m_i}\} \cup \{v'''_{l_1+1}, \ldots, v'''_{l_2}\}$$

and the vertices $v'''_1, \ldots, v'''_l$ are connected to the subgraphs $L_1, \ldots, L_n$ in a way the line graph $L(G)$ is connected (See Figure 3).

So, without loss of generality, the edge subsets of G in Equation 1 are $E_i = \{v_{i,1}v_{i,2}, \ldots, v_{i,1}v_{i,m_i+1}\}$ if $1 \leq i \leq h$ and $\{v_{i,1}v_{i,2}, v_{i,3}v_{i,4}, \ldots, v_{i,2m_i-1}v_{i,2m_i}\}$ if $h+1 \leq i \leq n$, and $F = F_1 \cup F_2 \cup F_3$, where $F_1 \subseteq \bigcup_{1 \leq i \neq j \leq h} \{v_{i,1}v_{j,s} | 2 \leq s \leq n_j\}$,

$$F_2 \subseteq \bigcup_{h+1 \leq i \neq j \leq n} \{v_{i,r}v_{j,s} | 2m_i + 1 \leq r \leq n_i, 2m_j + 1 \leq s \leq n_j\}$$

and

$$F_3 \subseteq \bigcup_{1 \leq i \leq h, h+1 \leq j \leq n} \{v_{i,r}v_{j,s} | 2 \leq r \leq n_i, 1 \leq s \leq n_j\}$$

(See Figure 4). So, G is a combination of the graphs of types (I) and (II).

Conversely, if the graph G is a combination of the graphs $H_1, \ldots, H_m$ of type (I) and the graphs $K_1, \ldots, K_n$ of type (II), then the union of $\{c_{H_1}u_{H_1}, \ldots, c_{H_m}u_{H_m}\}$, $\bigcup_{1 \leq i \leq n} \{uv | \{u, v\} \in W(k_i)\}$ and

$$\bigcup_{1 \leq i \neq j \leq n} \{uv \in E(L(G)) | u \in V(K_i) \setminus \cup W(K_i), v \in V(K_j) \setminus \cup W(K_j)\}$$

forms a 1-perfect code for the line graph $L(G)$ and so, the proof is completed.

4 Conclusion

In this paper, we determined all of the graphs G that both them and their line graphs contain 1-perfect codes. In general, we have the following open problem which is answered in the case $s = t = 1$:

Open problem. Let $s, t \geq 1$. Determine all of the graphs G that both G and its line graph $L(G)$ contain s- and t-perfect codes, respectively.

Appendix

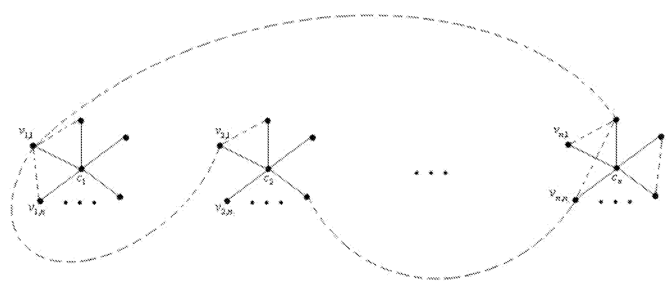

Figure 1: The graph G containing a 1-perfect code.

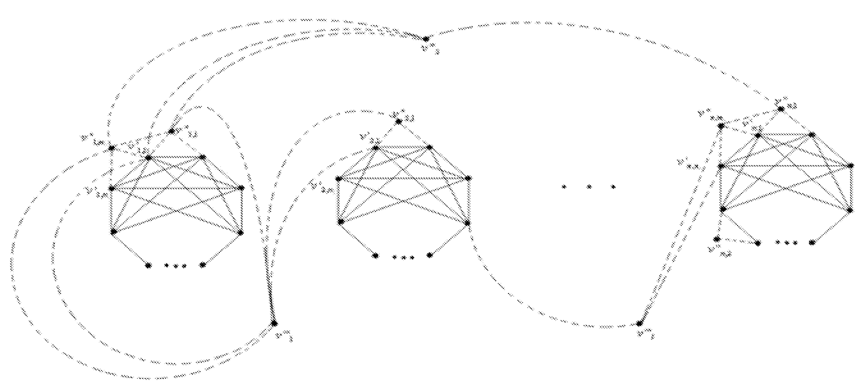

Figure 2: The line graph $L(G)$, where G contains a 1-perfect code.

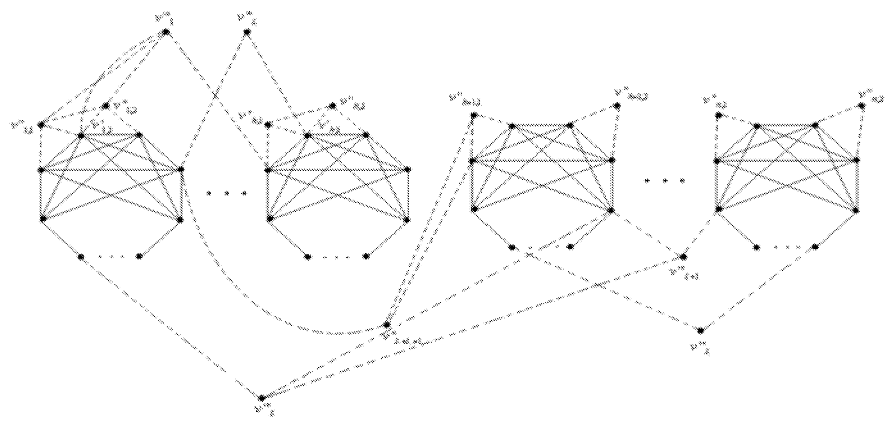

Figure 3: The graph $L(G)$, where G and $L(G)$ contain 1-perfect codes.

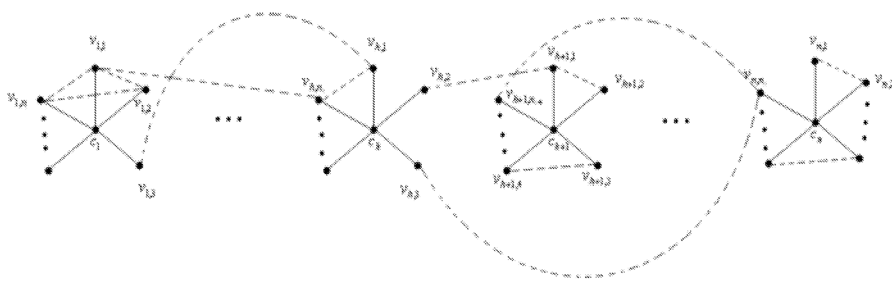

Figure 4: The graph G, where G and $L(G)$ contain 1-perfect codes.

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