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Special M-Hyperidentities in Triregular Leftmost without Loop and Reverse Arc Graph Varieties of Type (2,0)

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Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A class of graph algebras V is called a graph variety if $V = Mod_g\Sigma$ where Σ is a subset of $T(X) \times T(X)$. A graph variety $V' = Mod_g\Sigma^{'}$ is called a triregular leftmost without loop and reverse arc graph variety if $\Sigma^{'}$ is a set of triregular leftmost without loop and reverse arc term equations. A term equation $s \approx t$ is called an identity in a variety V if $\underline{A(G)}$ satisfies $s \approx t$ for all $G \in V$. An identity $s \approx t$ of a variety V is called a hyperidentity of a graph algebra $\underline{A(G)}$, $G \in V$ whenever the operation symbols occuring in s and t are replaced by any term operations of $\underline{A(G)}$ of the appropriate arity, the resulting identities hold in $\underline{A(G)}$. An identity $s \approx t$ of a variety V is called an M-hyperidentity of a graph algebra $\underline{A(G)}$, $G \in V$ whenever the operation symbols occuring in s and t are replaced by any term operations in a subgroupoid M of term operations of $\underline{A(G)}$ of the appropriate arity, the resulting identities hold in $\underline{A(G)}$.

In this paper we characterize special M-hyperidentities in each triregular leftmost without loop and reverse arc graph variety. For identities, varieties and other basic concepts of universal algebra see e.g. in [4].

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1 Introduction.

An identity $s \approx t$ of terms s, t of any type τ is called a *hyperidentity* (*M-hyperidentity*) of an algebra $\underline{A}$ if whenever the operation symbols occurring in s and t are replaced by any term operations (any term operations in a subgroupoid M of term operations) of $\underline{A}$ of the appropriate arity, the resulting identity holds in $\underline{A}$. Hyperidentities can be defined more precisely by using the concept of a hypersubstitution, which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in [6].

We fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is $n_i - ary$. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every $n_i - ary$ operation symbol f_i an $n_i - ary$ term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). By $\hat{\sigma}$ we denote the extension of the hypersubstitution σ to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Graph algebras have been invented in [13] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In [12], graph varieties had been investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [11], these investigations are extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by identities for their corresponding graph algebras. The answer is a theorem of

Birkhoff-type, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

In [1], Apinant Ananpinitwatna and Tiang Poomsa-ard characterized special M-hyperidentity in all biregular leftmost graph varieties of type (2,0). In [2], Apinant Ananpinitwatna and Tiang Poomsa-ard characterized special M-hyperidentity in all (x(yz))z with loop graph varieties of type (2,0). In [15], M. Thongmoon and T. Poomsa-ard characterized all triregular leftmost without loop and reverse arc graph varieties of type (2,0). In [3], R. Butkote and T. Poomsa-ard characterized identities in each triregular leftmost without loop and reverse arc graph variety of type (2,0). In [15], M. Thongmoon and T. Poomsa-ard characterized hyperidentities in each triregular leftmost without loop and reverse arc graph variety of type (2,0).

In this paper we characterize special M-hyperidentities in each triregular leftmost without loop and reverse arc graph variety of type (2,0).

2 Terms, identities and graph varieties.

Dealing with terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞, too).

Definition 2.1. The set $T(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...$, and ∞ are terms;
(ii) if t_1 and t_2 are terms, then t_1t_2 is a term;
(iii) $T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$ and rightmost variable of a term t is denoted by $R(t)$. A term, in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1 t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. A non-trivial term t of type $\tau = (2, 0)$ is called a term without loop and reverse arc if and only if for any $x \in V(t)$, $(x, x) \notin E(t)$ and for any $x, y \in V(t)$ with $x \neq y$ if $(x, y) \in E(t)$, then $(y, x) \notin E(t)$. A term equation $s \approx t$ of type $\tau = (2, 0)$ is called triregular leftmost without loop and reverse arc term equation if and only if $V(s) = V(t)$, $L(s) = L(t)$, $|V(s)| = 3$ and s, t are terms without loop and reverse arc.

Definition 2.4. We say that a graph $G = (V, E)$ *satisfies an identity* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e. we have $s = t$ for every assignment $V(s) \cup V(t) \to V\overline{\cup\{\infty\}}$), and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of identities (i.e., $\Sigma \subset T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$, $\quad \mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,

$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$, $\quad Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t,$

$Mod_g \Sigma = \{G \mid G \text{ is a graph and } G \models \Sigma\}$, $\quad \mathcal{V}_g(\mathcal{G}) = Mod_g Id\mathcal{G}$.

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of identities such that $\mathcal{G} = \mathrm{Mod}_g \Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

3 Triregular leftmost without loop and reverse arc graph varieties and identities.

In [15] M. Thongmoon and T. Poomsa-ard characterized all triregular leftmost without loop and reverse arc graph varieties as the following.

$\mathcal{K}_0 = Mod\{x(yz) \approx x(yz)\}$, $\quad \mathcal{K}_1 = Mod\{x(yz) \approx x(zy)\}$
$\mathcal{K}_2 = Mod\{x(yz) \approx (xy)z\}$, $\quad \mathcal{K}_3 = Mod\{x(yz) \approx x(y(zx))\}$
$\mathcal{K}_4 = Mod\{x(yz) \approx x(z(yx))\}$, $\quad \mathcal{K}_5 = Mod\{x(yz) \approx (x(yz))z\}$
$\mathcal{K}_6 = Mod\{(xy)z \approx x(y(zx))\}$, $\quad \mathcal{K}_7 = Mod\{(xy)z \approx (x(yz))z\}$
$\mathcal{K}_8 = Mod\{x(y(zx)) \approx x(z(yx))\}$, $\quad \mathcal{K}_9 = Mod\{x(y(zx)) \approx (x(yz))z\}$
$\mathcal{K}_{10} = Mod\{(x(yz))z \approx (xy)(zy)\}$, $\quad \mathcal{K}_{11} = \mathcal{K}_5 \cap \mathcal{K}_{10}$, $\quad \mathcal{K}_{12} = \mathcal{K}_8 \cap \mathcal{K}_{10}$.

In [3] R. Butkote and T. Poomsa-ard characterized identities in each triregular leftmost without loop and reverse arc graph variety. The common properties of an identity $s \approx t$ in each triregular leftmost without loop and reverse arc graph variety are (i) $L(s) = L(t)$, (ii) $V(s) = V(t)$. Clearly, if $s \approx t$ is a trivial equation (s,t are trivial or $G(s) = G(t)$ and, $L(s) = L(t)$), then $s \approx t$ is an identity in each triregular leftmost without loop and reverse arc graph variety. Further, if s is a trivial term and t is a non-trivial term or both of them are non-trivial with $L(s) \neq L(t)$ or $V(s) \neq V(t)$, then $s \approx t$ is not an identity in every triregular leftmost without loop and reverse arc graph variety, since for a complete graph G with more than one vertex, we have an evaluation of the variables h such that $h(s) = \infty$ and $h(t) \neq \infty$. Hence, we consider the case that $s \approx t$ is a non-trivial equation with $G(s) \neq G(t)$, $V(s) = V(t)$ and $L(s) = L(t)$. For short, we will quote only which we need to referent. Before we do this let us introduce some notation. For any non-trivial term t, $x \in V(t)$ and for any $(x, y) \in E(t)$ with $x \neq y$, let

$N_i^t(x) = \{x' \in V(t) \mid x' \text{ is an in-neighbor of } x \text{ in } G(t)\}$,
$N_o^t(x) = \{x' \in V(t) \mid x' \text{ is an out-neighbor of } x \text{ in } G(t)\}$,
$A_x(t) = \{x' \in V(t) \mid x' = x \text{ or there exists a dipath from } x \text{ to } x' \text{ in } G(t)\}$,
$A_x^{'}(t) = \{x' \in V(t) \mid x' = x \text{ or there exists a dipath from } x' \text{ to } x \text{ in } G(t)\}$,
$A_x^*(t) = A_x(t) \cup A_x^{'}(t)$, $B_x(t) = \bigcup\limits_{x' \in A_x^*(t)} A_{x'}(t)$, $B_x^{'}(t) = \bigcup\limits_{x' \in A_x^*(t)} A_{x'}^{'}(t)$.

$A^0_{(x,y)}(t) = \{(x, y)\}$, $A^1_{(x,y)}(t)$ be the set of edges $(u, x), (y, u) \in E(t)$ for some $u \in V(s)$, $A^2_{(x,y)}(t)$ be the set of edge $(u', v') \in E(t)$ whenever $(u', v') \in A^1_{(u,v)}(t)$ for some $(u, v) \in A^1_{(x,y)}(t)$,...,$A^n_{(x,y)}(t)$ be the set of edge $(u^{''}, v^{''}) \in E(t)$ whenever $(u^{''}, v^{''}) \in A^1_{(u,v)}(t)$ for some $(u, v) \in A^{n-1}_{(x,y)}(t)$. Let $A^*_{(x,y)}(t) = \bigcup\limits_{i=0}^{\infty} A^i_{(x,y)}(t)$.

The identities in each triregular without loop and reverse arc graph variety was characterized in each [3] as the following table:

Table 2. Triregular leftmost without loop and reverse arc graph varieties and the property of terms s and t.

Variety	Property of s and t
$\mathcal{K}_1$	(i) $N_i^s(L(s)) \neq \phi$ if and only if $N_i^t(L(t)) \neq \phi$, (ii) if $N_i^s(L(s)) = \phi$, then (a) for any $x \in V(s)$ there exist $y, z \in V(s)$ such that $(z, y), (y, x) \in E(s)$ if and only if there exist $y', z' \in V(s)$ such that $(z', y'), (y', x) \in E(t)$ or there exist $u', v' \in V(s)$ such that $(u', x), (x, v') \in E(t)$, (b) for any $x, y \in V(s)$ with $x \neq y$, $(B_x(s) \cap B_y(s)) \neq \phi$ or $(B'_x(s) \cap B'_y(s)) - \{L(s)\} \neq \phi$ if and only if $(B_x(t) \cap B_y(t))$ $\neq \phi$ or $(B'_x(t) \cap B'_y(t)) - \{L(t)\} \neq \phi$.
$\mathcal{K}_5$	(i) for any $x \in V(s)$, there exists $y \in V(s)$ such that $(x, y), (y, x) \in E(s)$ if and only if there exists $y' \in V(s)$ such that $(x, y'), (y', x) \in E(t)$, (ii) for any $x, y \in V(s)$ with $x \neq y$, $y \in A_x(s)$ if and only if $y \in A_x(t)$.
$\mathcal{K}_8$	(i) for any $x \in V(s)$, $(x, x) \in E(s)$ if and only if $(x, x) \in E(t)$, (ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or there exists $z \in V(s)$ with $z \neq x,\ z \neq y$ such that $(y, x), (x, z), (z, y) \in E(s)$ if and only if $(x, y) \in E(t)$ or there exists $z' \in V(s)$ with $z' \neq x,\ z' \neq y$ such that $(y, x), (x, z'), (z', y) \in E(t)$.
$\mathcal{K}_9$	(i) for any $x \in V(s)$, $(x, x) \in E(s)$ if and only if $(x, x) \in E(t)$, (ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and, $(x, x) \in E(s)$ or $(y, y) \in E(s)$ or there exists $z \in V(s)$, $z \neq x$, $z \neq y$ such that z is an in-neighbor or an out -neighbor both of x and y in $G(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and, $(x, x) \in E(t)$ or $(y, y) \in E(t)$ or there exists $z' \in V(s)$, $z' \neq x$, $z' \neq y$ such that z' is an in-neighbor or an out-neighbor both of x and y in $G(t)$.
$\mathcal{K}_{10}$	(i) for any $x \in V(s)$, $(x, x) \in E(s)$ if and only if $(x, x) \in E(t)$, (ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and, there exists $(u, v) \in A^*_{(y,x)}(s)$ such that $(u, u) \in E(s)$ or there exists $w \in V(s)$ such that $(w, u), (w, v) \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and, there exists $(u', v') \in A^*_{(y,x)}(t)$ such that $(u', u') \in E(t)$ or there exists $w' \in V(s)$ such that $(w', u'), (w', v') \in E(t)$.

Table 2. (*Continued*)

Variety	Property of s and t
$\mathcal{K}_{11}$	(i) for any $x \in V(s)$, there exist $u, v \in V(s)$ such that (u, x), $(x, v) \in E(s)$ or $(u, v), (v, x) \in E(s)$ if and only if there exist $u', v' \in V(s)$ such that $(u', x), (x, v') \in E(t)$ or $(u', v'), (v', x)$ $\in E(t)$, (ii) for any $x, y \in V(s)$, $x \neq y$, $y \in A_x(s)$ or $N_i^s(y) \neq \phi$ and $x \in A_y(s)$ if and only if $y \in A_x(t)$ or $N_i^t(y) \neq \phi$ and $x \in A_y(t)$.

4 Hypersubstitution and proper hypersubstitution

Let $\mathcal{K}$ be a graph variety. Now we want to formulate precisely the concept of a graph hypersubstitution for graph algebras.

Definition 4.1. A mapping $\sigma : \{f, \infty\} \to T(X_2)$, where $X_2 = \{x_1, x_2\}$ and f is the operation symbol corresponding to the binary operation of a graph algebra is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in T(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $\mathcal{K}$ *graph hyperidentity* iff for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{K}$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{K}$ we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions.

In [8], the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{K}$-*equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an idetity in $\mathcal{K}$. In this case we write $\sigma_1 \sim_{\mathcal{K}} \sigma_2$.

The following lemma was proved in [9].

Lemma 4.1. *If* $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}$ *and* $\sigma_1 \sim_{\mathcal{K}} \sigma_2$ *then,* $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}$.

Therefore, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{K}}$.

In [10], it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Now, we want to describe how to construct the normal form term. Let t be a non-trivial term. The *normal form term* of t is the term $NF(t)$ constructed by the following algorithm:

(i) Construct $G(t) = (V(t), E(t))$.

(ii) Construct for every $x \in V(t)$ the list $l_x = (x_{i_1}, ..., x_{i_{k(x)}})$ of all out-neighbors (i.e. $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$) ordered by increasing indices $i_1 \leq ... \leq i_{k(x)}$ and let s_x be the term $(...((xx_{i_1})x_{i_2})...x_{i_{k(x)}})$.

(iii) Starting with $x := L(t), Z := V(t), s := L(t)$, choose the variable $x_i \in Z \cap V(s)$ with the least index i, substitute the first occurrence of x_i by the term s_{x_i}, denote the resulting term again by s and put $Z := Z \setminus \{x_i\}$. While $Z \neq \phi$ continue this procedure. The resulting term is the normal form $NF(t)$.

The algorithm stops after a finite number of steps, since $G(t)$ is a rooted graph. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

The following definition was given in [5].

Definition 4.4. The graph hypersubstitution $\sigma_{NF(t)}$, is called *normal form graph hypersubstitution*. Here $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}$. Then for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in T(X_2)$, one obtains $\sigma_t \sim_{\mathcal{K}} \sigma_{NF(t)}$.

In [5], all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the Table 2.

Table 2. normal form terms

normal form term	graph hypers	normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

Let $M_{\mathcal{G}}$ be the set of all normal form graph hypersubstitutions. Then we get,

$$M_{\mathcal{G}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}.$$

The concept of a proper hypersubstitution of a class of algebras was introduced in [9].

Definition 4.5. A hypersubstitution σ is called *proper with respect to a class $\mathcal{K}$ of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

The following lemma was proved in [5].

Lemma 4.2. *For each non-trivial term $s, (s \neq x \in X)$ and for all $u, v \in X$, we have*

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u)|(u,v) \in E(s)\},$$

$$E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v)|(u,v) \in E(s)\},$$

and

$$E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u)|(u,v) \in E(s)\}.$$

By the similar way we prove that,

$$E(\hat{\sigma}_{10}[s]) = E(s) \cup \{(u,u),(v,v)|(u,v) \in E(s)\}.$$

Let $PM_{\mathcal{K}}$ be the set of all proper graph hypersubstitutions with respect to the class $\mathcal{K}$. In [14] it was found that:
$PM_{\mathcal{K}_0} = PM_{\mathcal{K}_8} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$.
$PM_{\mathcal{K}_1} = PM_{\mathcal{K}_{11}} = \{\sigma_0, \sigma_6, \sigma_8\}$.
$PM_{\mathcal{K}_2} = PM_{\mathcal{K}_3} = PM_{\mathcal{K}_4} = PM_{\mathcal{K}_7} = \{\sigma_0, \sigma_6\}$.
$PM_{\mathcal{K}_6} = \{\sigma_0\}$. $PM_{\mathcal{K}_9} = \{\sigma_0, \sigma_{10}, \sigma_{12}\}$.
$PM_{\mathcal{K}_{10}} = PM_{\mathcal{K}_{12}}\{\sigma_0, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{16}\}$.

5 Special M-hyperidentities

We know that a graph identity $s \approx t$ is a graph hyperidentity, if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is a graph identity for all $\sigma \in M_{\mathcal{G}}$. Let M be a subgroupoid of $M_{\mathcal{G}}$. Then, a graph identity $s \approx t$ is an M-graph hyperidentity (M-hyperidentity), if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is a graph identity for all $\sigma \in M$. In [4], K. Denecke and S.L. Wismath defined special subgroupoid of M_g as the following.

Definition 5.1. *(i) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be leftmost if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ is x_1. Let $Left(\tau)$ be the set of all leftmost hypersubstitutions of type τ.*
(ii) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be outermost if for every $i \in I$, the first variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ is x_1 and the last variable is x_{n_i}. Let $Out(\tau)$ be the set of all outermost hypersubstitutions of type τ.
(iii) A hypersubstitution $\sigma \in Hyp(\tau)$ is said to be rightmost if for every $i \in I$, the last variable in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$ is x_{n_i}. Let $Right(\tau)$ be the set of all rightmost hypersubstitutions of type τ. Note that $Out(\tau) = Right(\tau) \cap Left(\tau)$.

(iv) A hypersubstitution $\sigma \in Hyp(\tau)$ is called regular if for every $i \in I$, each of the variables $x_1, ..., x_{n_i}$ occurs in $\hat{\sigma}[f_i(x_1, ..., x_{n_i})]$. Let $Reg(\tau)$ be the set of all regular hypersubstitutions of type τ.
(v) A hypersubstitution $\sigma \in Hyp(\tau)$ is called symmetrical if for every $i \in I$, there is a permutation s_i on the set $\{1, ..., n_i\}$ such that $\hat{\sigma}[f_i(x_1, ..., x_{n_i})] = f_i(x_{s_i(1)}, ..., x_{s_i(n_i)})$. Let $D(\tau)$ be the set of all symmetrical hypersubstitutions of type τ.
(vi) We will call a hypersubstitution σ of type τ a pre-hypersubstitution if for every $i \in I$, the term $\sigma(f_i)$ is not a variable. Let $Pre(\tau)$ be the set of all pre-hypersubstitutions of type τ.

From Definition 5.1, we have:
$M_{Left} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$.
$M_{Right} = \{\sigma_0, \sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.
$M_{Out} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$.
$M_{Reg} = \{\sigma_0, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.
$M_D = \{\sigma_0, \sigma_5\}$.
$M_{Pre} = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$.

Definition 5.2. *Let V be a graph variety of type τ, and let $s \approx t$ be an identity of V. Let M be a subgroupoid of $Hyp(\tau)$. Then $s \approx t$ is called an M-hyperidentity with respect to V, if for every $\sigma \in M$, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V.*

For any triregular leftmost without loop and reverse arc graph variety $\mathcal{K}$ and for any $s \approx t \in Id\mathcal{K}$. We want to characterize the property of s and t such that $s \approx t$ is an M_{Left}-hyperidentity, M_{Right}-hyperidentity, M_{Out}-hyperidentity, M_{Reg}-hyperidentity, M_D-hyperidentity and M_{Pre}-hyperidentity with respect to $\mathcal{K}$ for all triregular leftmost without loop and reverse arc graph varieties $\mathcal{K}$.

At first we consider the M_D-hyperidentity. Since $M_D = \{\sigma_0, \sigma_5\}$, let $\mathcal{K}$ be any triregular leftmost without loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if $s \approx t$ is a trivial term equation, then $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$. For the case $s \approx t$ is a non-trivial equation, we have $s \approx t$ is an M_D-hyperidentity with respect to $\mathcal{K}$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}$.

For M_{Left}-hyperidentity. Since $M_{Left} = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$, let $\mathcal{K}$ be any triregular leftmost without loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if $s \approx t$ is a trivial term equation, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}$ if and only if $L(s) = L(t)$. Now we consider the case $s \approx t$ is a non-trivial equation. We characterize M_{Left}-hyperidentity with respect to all triregular leftmost without loop and reverse arc graph varieties as the following theorems:

Theorem 5.1. *Let $s \approx t$ be a non-trivial equation and let $\mathcal{K}_i$, $i \in \{0, 1, 2, ..., 11\}$ be triregular leftmost without loop and reverse arc graph varieties. If $s \approx t \in Id\mathcal{K}_i$, then $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_1$. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. Since $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$ and $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$, we have $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \in Id\mathcal{K}_1$. Since $\sigma_6 \sim_{\mathcal{K}_1} \sigma_{10} \sim_{\mathcal{K}_1} \sigma_{12} \sim_{\mathcal{K}_1} \sigma_{14} \sim_{\mathcal{K}_1} \sigma_{16} \sim_{\mathcal{K}_1} \sigma_{18}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$. Hence, $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_1$. The proof of other graph varieties is similar to the proof of $\mathcal{K}_i$. □

Theorem 5.2. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_9$. Then, $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_9$ if and only if $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_9$.*

Proof. If $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_9$, then $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t] \in Id\mathcal{K}_9$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_9$ and that $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, and $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$ are also identities in $\mathcal{K}_9$, too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Left}. If $\sigma \in \{\sigma_0, \sigma_{10}, \sigma_{12}\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$. By assumption, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$ and $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$ are also identities in $\mathcal{K}_9$. Since $\hat{\sigma}_1[s] = L(s) = L(t) = \hat{\sigma}_1[t]$ and $\hat{\sigma}_3[s] = L(s)L(s) = L(t)L(t) = \hat{\sigma}_3[t]$, we have $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \in Id\mathcal{K}_9$. Since $\sigma_6 \sim_{\mathcal{K}_9} \sigma_{14}$, $\sigma_8 \sim_{\mathcal{K}_9} \sigma_{16}$ and $\sigma_{10} \sim_{\mathcal{K}_9} \sigma_{18}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$ for all $\sigma \in \{\sigma_{14}, \sigma_{16}, \sigma_{18}\}$. Hence, $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_9$. □

Theorem 5.3. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_i$ $i =$ 10, 12. Then, $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_i$.*

Proof. The proof is similar to the proof of Theorem 5.2. □

For M_{Out}-hyperidentity. Since $M_{Out} = \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{16}, \sigma_{18}\}$, let $\mathcal{K}$ be any triregular leftmost without loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if $s \approx t$ is a trivial term equation, then $s \approx t$ is an M_{Out}-hyperidentity with respect to $\mathcal{K}$. For the case $s \approx t$ is a non-trivial equation, since $M_{Out} \subset M_{Left}$, so we can check that it has the same results as M_{Left}-hyperidentity.

For M_{Reg}-hyperidentity. Since $M_{Reg} = \{\sigma_0, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, let $\mathcal{K}$ be any triregular leftmost without loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if $s \approx t$ is a trivial term equathion, then $s \approx t$ is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$. For the case $s \approx t$ is a non-trivial equation. We get the same result as hyperidentity which we can see the prove in [14].

Theorem 5.4. *An identity $s \approx t$ in $\mathcal{K} \in \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_8, \mathcal{K}_{11}\}$, where $s \approx t$ is a non-trivial equation is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$ is also an identity in $\mathcal{K}$.*

Theorem 5.5. *An identity $s \approx t$ in $\mathcal{K}_9$, where $s \approx t$ is a non-trivial equation is an M_{Reg}-hyperidentity with respect to $\mathcal{K}_9$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$ and $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ are also identities in $\mathcal{K}_9$.*

Theorem 5.6. *An identity $s \approx t$ in $\mathcal{K} \in \{\mathcal{K}_{10}, \mathcal{K}_{12}\}$, where $s \approx t$ is a non-trivial equation is an M_{Reg}-hyperidentity with respect to $\mathcal{K}$ if and only if $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$ and $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$ are also identities in $\mathcal{K}$.*

For M_{Pre}-hyperidentity. Since $M_{Pre} = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, let $\mathcal{K}$ be any triregular leftmost without loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if $s \approx t$ is a trivial term equation, then $s \approx t$ is an M_{Pre}-hyperidentity with respect to $\mathcal{K}$ if and only if s and t have the same leftmost variable and the same rightmost variable. For the case $s \approx t$ is non-trivial equation, since $M_{Reg} = M_{Pre} - \{\sigma_3, \sigma_4\}$, we have the same results as M_{Reg}-hyperidentity.

For M_{Right}-hyperidentity. Since $M_{Right} = \{\sigma_0, \sigma_2, \sigma_4, \sigma_6, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$, let $\mathcal{K}$ be any triregular leftmost without loop and reverse arc graph variety and for any $s \approx t \in Id\mathcal{K}$. We see that if $s \approx t$ is a trivial term equation, then $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}$ if and only if they have the same rightmost variables. So, we will consider the case $s \approx t$ is non-trivial equation. We characterize M_{Right}-hyperidentity with respect to all triregular leftmost without loop and reverse arc graph varieties as the following theorems:

Theorem 5.7. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_i$, $i = 0, 8$ Then, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are also identities in $\mathcal{K}_i$.*

Proof. Consider for $\hat{\sigma}_8$, let $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_8$. We have $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are also identities in $\mathcal{K}_i$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_8$ and that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$,$\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are also identities in $\mathcal{K}_8$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_8$. By assumption, $\mathcal{K}_8$ and that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are also identities in $\mathcal{K}_8$. Hence, $R(s) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_7[t]) = R(t)$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_8$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_8$. Since $\sigma_{10} \circ_N \sigma_7 = \sigma_{11}$, $\sigma_{12} \circ_N \sigma_7 = \sigma_{15}$, $\sigma_{12} \circ_N \sigma_{11} = \sigma_{19}$ and $\hat{\sigma}_{10}$, $\hat{\sigma}_{12}$ are proper hypersubstitution, we have that $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$, $\hat{\sigma}_{15}[s] \approx \hat{\sigma}_{15}[t]$ and $\hat{\sigma}_{19}[s] \approx$

$\hat{\sigma}_{19}[t]$ are identities in $\mathcal{K}_8$. Hence, $s \approx t$ is an M_{right}-hyperidentity with respect to $\mathcal{K}_2$. The proof of $\mathcal{K}_0$ graph variety is similar to the proof of $\mathcal{K}_8$. □

Theorem 5.8. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_i$, $i = 1, 11$ Then, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_1$, let $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_1$. We have $\hat{\sigma_7}[s] \approx \hat{\sigma_7}[t] \in Id\mathcal{K}_1$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_1$ and that $\hat{\sigma_7}[s] \approx \hat{\sigma_7}[t] \in Id\mathcal{K}_1$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$. By assumption, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$ is also an identity in $\mathcal{K}_1$. Hence, $R(s) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_7[t]) = R(t)$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_1$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_1$. Since $\sigma_6 {\sim_{\mathcal{K}_1}} \sigma_{10} {\sim_{\mathcal{K}_1}} \sigma_{16} {\sim_{\mathcal{K}_1}} \sigma_{18}$ and $\sigma_7 {\sim_{\mathcal{K}_1}} \sigma_{11} {\sim_{\mathcal{K}_1}} \sigma_{13} {\sim_{\mathcal{K}_1}} \sigma_{15} {\sim_{\mathcal{K}_1}} \sigma_{17} {\sim_{\mathcal{K}_1}} \sigma_{19}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_1$ for all $\sigma \in \{\sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. Hence, $s \approx t$ is an M_{Left}-hyperidentity with respect to $\mathcal{K}_1$. The proof of $\mathcal{K}_{11}$ graph variety is similar to the proof of $\mathcal{K}_1$. □

Theorem 5.9. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_i$, $i = 2, 3, 4, 7$ Then, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_3$, let $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_1$. We have $\hat{\sigma_7}[s] \approx \hat{\sigma_7}[t] \in Id\mathcal{K}_3$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_3$ and that $\hat{\sigma_7}[s] \approx \hat{\sigma_7}[t] \in Id\mathcal{K}_3$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_6\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_3$. By assumption, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$ is also an identity in $\mathcal{K}_3$. Hence, $R(s) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_7[t]) = R(t)$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_3$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_3$. Since $\sigma_6 {\sim_{\mathcal{K}_3}} \sigma_8 {\sim_{\mathcal{K}_3}} \sigma_{10} {\sim_{\mathcal{K}_3}} \sigma_{16} {\sim_{\mathcal{K}_3}} \sigma_{18}$ and $\sigma_7 {\sim_{\mathcal{K}_3}} \sigma_{11} {\sim_{\mathcal{K}_3}} \sigma_{13} {\sim_{\mathcal{K}_3}} \sigma_{15} {\sim_{\mathcal{K}_3}} \sigma_{17} {\sim_{\mathcal{K}_3}} \sigma_{19}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_3$ for all $\sigma \in \{\sigma_8, \sigma_{10}, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. Hence, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_3$. The proof of other graph varieties are similar to the proof of $\mathcal{K}_3$. □

Theorem 5.10. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_5$. Then, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_5$ if and only if $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_5$.*

Proof. If $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_5$, then $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_5$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_5$ and that $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$ is also an identity in $\mathcal{K}_5$, too. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$, then σ is a proper hypersubstitution. Hence, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_5$. By assumption, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$ is also an identity in $\mathcal{K}_5$. Hence, $\hat{\sigma}_7[s] = R(s) = R(t) = \hat{\sigma}_7[t]$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_5$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_5$. Since $\sigma_{10} \circ_N \sigma_7 = \sigma_{11}$, $\sigma_{12} \circ_N \sigma_7 = \sigma_{15}$ and $\hat{\sigma}_{10}$, $\hat{\sigma}_{12}$ are proper hypersubstitution, we have that $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$ and $\hat{\sigma}_{15}[s] \approx \hat{\sigma}_{15}[t]$ are identities in $\mathcal{K}_5$. Since $\sigma_{12} \sim_{\mathcal{K}_5} \sigma_{16} \sim_{\mathcal{K}_5} \sigma_{18}$ and $\sigma_{13} \sim_{\mathcal{K}_5} \sigma_{15} \sim_{\mathcal{K}_5} \sigma_{17} \sim_{\mathcal{K}_5} \sigma_{19}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_5$ for all $\sigma \in \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. Hence, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_5$. □

Theorem 5.11. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_9$. Then, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_9$ if and only if $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ are also identities in $Id\mathcal{K}_9$.*

Proof. Let $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_9$. Then, we have $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ are also identities in $Id\mathcal{K}_9$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_9$ and that $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ are also identities in $Id\mathcal{K}_9$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_{10}, \sigma_{12}\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$. By assumption, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_8[s] \approx \hat{\sigma}_8[t]$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t]$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ are identities in $Id\mathcal{K}_9$. Hence, $R(s) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_7[t]) = R(t)$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_9$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_9$. Since $\sigma_{10} \circ_N \sigma_7 = \sigma_{11}$ and $\hat{\sigma}_{10}$ is a proper hypersbstitution, we have that $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t] \in Id\mathcal{K}_9$. Since $\sigma_7 \sim_{\mathcal{K}_9} \sigma_{15}$, $\sigma_8 \sim_{\mathcal{K}_9} \sigma_{16}$, $\sigma_9 \sim_{\mathcal{K}_9} \sigma_{17}$, $\sigma_{10} \sim_{\mathcal{K}_{10}} \sigma_{18}$ and $\sigma_{11} \sim_{\mathcal{K}_{10}} \sigma_{19}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_9$ for all $\sigma \in \{\sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}\}$. Hence, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_9$. □

Theorem 5.12. *Let $s \approx t$ be a non-trivial equation and let $s \approx t \in Id\mathcal{K}_i$ $i = 10, 12$. Then, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_i$ if and only if $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are identities in $\mathcal{K}_i$.*

Proof. Consider for $\mathcal{K}_{10}$, let $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{10}$. Then, we have $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are identities in $\mathcal{K}_{10}$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_{10}$ and that $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are also identities in $\mathcal{K}_{10}$. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from M_{Right}. If $\sigma \in \{\sigma_0, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{16}\}$, then σ is a proper hypersubstitution. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{10}$. By assumption, $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t]$, $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t]$ and $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t]$ are identities in $\mathcal{K}_{10}$. Hence,

$R(s) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_7[t]) = R(t)$. Since $\hat{\sigma}_2[s] = R(s) = R(t) = \hat{\sigma}_2[t]$ and $\hat{\sigma}_4[s] = R(s)R(s) = R(t)R(t) = \hat{\sigma}_4[t]$, we have $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}_{10}$ and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \in Id\mathcal{K}_{10}$. Since $\sigma_{10} \circ_N \sigma_7 = \sigma_{11}$ and $\hat{\sigma}_{10}$ is a proper, we have that $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t]$ is an identity in $\mathcal{K}_{10}$. Since $\sigma_7 \sim_{\mathcal{K}_{10}} \sigma_{15}$, $\sigma_{10} \sim_{\mathcal{K}_{10}} \sigma_{18}$ and $\sigma_{11} \sim_{\mathcal{K}_{10}} \sigma_{19}$. We get that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{10}$ for all $\sigma \in \{\sigma_{15}, \sigma_{18}, \sigma_{19}\}$. Hence, $s \approx t$ is an M_{Right}-hyperidentity with respect to $\mathcal{K}_{10}$. The proof of $\mathcal{K}_{12}$ is similar to the proof of $\mathcal{K}_{10}$. $\square$

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