International Mathematical Forum, Vol. 7, 2012, no. 36, 1761 - 1766

Rings Radical over Subrings and Left FBN-ring

A.-R. M. Hassanein

Department of Mathematics, Faculty of Science
Al-Azhar University, Nasr City (11884), Cairo, Egypt
mhassanein_05@yahoo.com

Abstract

Let R be a ring with a subring A such that a power of every element of R lies in A. The following results are proved. If R is a semiprime left Noetherian ring and R is FBN-ring, so is A. If R is semiprime Noetherian rings, A is left FBN-ring and $J(A) \neq \{0\}$, then R is left FBN-ring. Different properties of R to be left fully bounded left Noetherian ring are studied. Furthermore, we show that if R is left Noetherian ring which is H-extension of a subring A, $A \subseteq Z(R)$, where $Z(R)$ is the center of R, then R is a left fulls bounde left Noetherian ring. Also, we show that if $R \underset{H}{|} A$ where A is a commutative subring of left Noetherian ring R. Then R is left fully bounded left Noetherian ring. Moreover, we proved that if R is weakly injective ring, so is A. Also, we show that if R is weakly R-injective, so then A is weakly A-injective.

Mathematics Subject Classification: Primary 16N50, 16N40, 16N60, 16P40, 16P50, 16P60 Secondary 16D50

Keywords: Rings radical over subrings, left fully bounded left Noetherian ring, semiprime left Golde ring, weakly injective ring

1 Introduction

Through this paper R will denote an associative ring and A will always denote a subring of R. Following Faith [2], we say R is radical over subring A, or R is A-radical if for each $r \in R$ there exists $n = n(r) \geq 1$ such that $r^n \in A$.

A left Noetherian ring R is left bounded if every essential left ideal I of R contains a nonzero two sided ideal of R. A left Noetherian ring is left fully bounded left Noetherian ring (called left FBN-ring) if evey prime factor ring of R is left bounded. A fully bounded Noetherian ring is left and right fully bounded Noetherian ring. Trivially, commutative Noetherian rings and finite algebras over them are fully bounded Noetherian rings. More generally, it is known [10] that a Noetherian ring is FBN-ring if it satisfies a polynomial identity.

Let R be a ring and let M and N be R-modules. Let $E(M)$ denote the injective hull of M. Recall that M is injective relative to N or simply N-injective if for each homomorphism.

$$\phi : N \to E(M) \qquad , \qquad \phi(N) \subset M.$$

This motivates the definition of weak relative-injectivity which is defined as follows:

An R-module M is said to be weakly-injective relative to the R-module N or weakly N-injective if for each R-homomorphism $\phi : N \to E(M)$, $\phi(N) \subset X \cong M$, for some submodule X of $E(M)$. A ring R is right weakly N-injective if the right module R is weakly N-injective [6].

An R-module M is called weakly R^n-injective if every n-element generated submodules of its injective hull $E(M)$ is contained in a submodule X of $E(M)$ isomorphic to M. An R-module M is called weakly-injective if it is weakly R^n injective for all $n > 0$. The ring R is called a right weakly-injective $(W-I)$ ring if R is weakly-injective as right R-module. In particular, a ring R is weakly R-injective (WRI) if R is weakly R-injective as right R-module.

2 Rings Radical over Subrings

In this section we establish a necessary and sufficient condition for a ring R which is radical over subring A to be left fully bounded left Noetherian ring.

Lemma 2.1. *Let R be a radical over subring A. Then for every ideal $I \lhd R$, $A \cap I$ is an ideal in A.*

Proof. It is clear that $A \cap I$ is a subring of A. Let $a \in A \cap I$ and $s \in A$, we want to show that $as, sa \in A \cap I$. Since $a \in A \cap I$, $s \in A$ hence $as, sa \in A$, $as, sa \in I$ $(a \subset R, I \lhd R)$. Therefore, $sa, as \in A \cap I$. Then $A \cap I$ is an ideal in A. □

Remark 2.1. From this Lemma, if R is radical over A and R is a Noetherian ring, then A is a Noetherian ring.

Lemma 2.2. *Let R be a radical over subring A, then for every prime ideal P in R, R/P is a radical over $A/P \cap A$.*

Proof. We want to show that for every $\bar{r} \in R/P$ $\exists$ $n(\bar{r})$ such that $\bar{r}^{n(\bar{r})} \in A/P \cap A$. It is clear that for every $\bar{r} \in R/P$, $\bar{r}^n = (r+p)^n = r^n + p$. but since, R is a radical over A and $A + P/P \simeq A/A \cap P$ this implies that $\bar{r}^n \in A/P \cap A$. Therefore R/P is radical over $A/P \cap A$. □

Theorem 2.3. *Let R be a semiprime left Noetherian ring, R is radical over subring A and R is a left FBN-ring. Then A is a semiprime left FBN-ring.*

Proof. Let P is a prime ideal of R. Passing to R/P we can assume without loss of generality that $P = 0$ and R is a prime Goldie.

Since R is a semprime left Goldie ring and R is radical over a subring A, then A is semiprime left Goldie ring by Theorem 4 in [3].

Assume that I is a left essential ideal of A, then by Zorn's lemma there exist a left maximal ideal I' of R with respect to $I' \cap I = 0$. therefore, $1 = I \oplus I'$ is an essential left ideal of R. Since R is a left FBN-ring, hence 1 contains a two-sided ideal $J \neq \{0\}$ of R. This implies that $\{0\} \neq J \cap A$ is a two-sided ideal of A and $\{0\} \neq J \cap A \subset L \cap A = I$.

Therefore, A is a left bounded and hence A is a left FBN-ring. □

Theorem 2.4. *Let R be a semiprime left Noetherian ring and R is radical over subring A. If A is left fully bounded left Noetherian ring and $J(A) \neq 0$. Then R is left fully bounded left Noetherian ring.*

Proof. We will show that every essential left ideal of a prime factor ring of R contains two-sided ideal $\neq \{0\}$ of R.

Suppose P is a prime ideal of R, passing to R/P, we can assume without loss of generality that $P = 0$, and R is a prime Goldie ring. Let I be a left essential ideal of R. Clearly $I' = I \cap A$ is essential left ideal of A by Remark 2 in [3]. Since A is left fully bounded left Noetherian ring, then I' contains a nonzero two-sided ideal of A, say $J \subset I'$. But since $J(A) \neq 0$, hence there is a central idempotent e of A such that $J(A) = eA$ [1].

Therefore, $eA \subset J \subset I'$ and by [4] $Z(A) \subset Z(R)$ consequently $e \in Z(R)$. Then $I \supset I' \supset J \supset eA$. Since I is a left essential ideal of R hence $I \supset eR \neq \{0\}$ where eR is a nonzero two-sided ideal of R generated by a central idempotent of R. Consequently R is a fully bounded left Noetherian ring. □

The following example show that, there is a left fully bounded left Noetherian ring A with no nonzero Jacobson radical $J(A)$.

Example 2.1. Let F be a field, then the formal power series $A = F[[x]]$ is a left fully bounded left Noetherian ring and $J(A) \neq \{0\}$.

Proof. Since F is a field hence the formal power series $F[[x]]$ is a Noetherian ring by [7]. But $F[[x]]$ is a commutative, therefore $F[[x]]$ is a left fully bounded left Noetherian ring. Also, $F[[x]]$ has unique maximal ideal $< x >$, hence $J(A) \neq \{0\}$. It is clear that any extension ring R of $A = F[[x]]$ is left fully bounded left Noetherian ring by Theorem 2.4. □

Using Theorem 1.1 in [2] we get the following result.

Theorem 2.5. *If R is a left Noetherian ring with no nil ideals $\neq \{0\}$, and if R is redical over a division subring $A \neq R$, then R is left fully bounded left Noetherian ring.*

Also, from Theorem 1.2 in [2], we deduce the following results.

Theorem 2.6. *If R is radical over commutative subring and if $J(R) = 0$. Then R is a left fully bounded left Noetherian ring.*

Corollary 2.7. *If R is left Noetherian semisimple ring and if n is fixed natural number such that to each pair $x, y \in R$, x is radical over the centralizer of y^N in R, then R is left fully bounded left Noetherian ring.*

Proof. From Corollary 1.4 in [2], we have R is a commutative ring. Therefore, R is left fully bounded left Noetherian ring. □

(*) For any $x, y \in R$, there exist natural numbers m, n such that $x^n y^m = y^m x^n$.

Theorem 2.8. *If R is a left Noetherian primitive ring with minimal left ideal, or if R is a subdirect sum of such rings, and if R satisfies (*), then R is a left fully bounded left Noetherian ring.*

Proof. Since R is a primitive ring with minimal left ideal, or R is a subdirect sum of such rings and if R satisfies (*). Then R is commutative from Theorem 1.5 in [2]. Then R is left fully bounded left Noetherian ring. □

If R is a left Noetherian ring with no nil ideals $\neq \{0\}$ and radical over a commutative (possibly semisimple) subring B in R necessarily left fully bounded left Noetherian ring?

If A is semisimple ideal, the answer is yes. This can be seen as follows: $J(A) = 0$ so that $J(R) \cap A = \{0\}$, whence $J(A)$ is nil. By hypothesis, $J(A) = \{0\}$ and R is left fully bounded left Noetherian ring by theorem 2.6.

3 H-Extension Ring

In this section we show that if $R|_H A$, where A commutative, then R is left fully bounded left Noetherian ring.

We begin with the following definitions.

Definition 3.1. A ring R is called as H-extension of a subring A denoted by $R|_H A$ if for every $r \in R$ there exists an integer $n(r) > 1$, such that $r^n - r \in A$ [8].

Theorem 3.1. *Let R be a left Noetherian ring which is H-extension of a subring A, $A \subseteq Z(R)$, where $Z(R)$ is the center of R, then R is a left fully bounded left Noetherian ring.*

Proof. Since R is an H-extension of a subring A, $A \subseteq Z(R)$, then R is an H-ring. hence R is a commutative (see [5], p. 317). Therefore, R is a left fully bounded left Noetherian ring. □

Theorem 3.2. *Suppose R is a left Noetherian radical ring without zero divisors and $R|_{_H} A$, where A is a commutative subring, then R is a left fully bounded left Noetherian ring.*

Proof. Since R is a radical ring without zero divisors and $R|_{_H} A$, A is a commutative ring, then R is a commutative ring by Proposition 4 [4]. Hence is a left fully bounded left Noetherian ring. □

Using Theorem 1 in [9], we deduce the following theorem.

Theorem 3.3. *Suppose $R|_{_H} A$ where A is a commutative subring of left Noetherian ring R. Then R is a left fully bounded left Noetherian ring.*

Also, from Theorem 2 in [9], we can get the following result.

Theorem 3.4. *Suppose R is a right Noetherian algebra over a filed F and $R|_{_H} A$, where A is a right ideal satisfying some identity. Then R is a left fully bounded left Noetherian ring.*

4 Weakly Injectivity

Theorem 4.1. *Let R be a radical over subring A and R is weakly-injective. Then A is weakly-injective.*

Proof. Let R is weakly injective ring, we will show that A is weakly-injective ring. Let $a_1, a_2, \cdots, a_n \in E(A_A) \subset E(R_R)$. Need $B \subseteq E(A_A)$ such that $a_1, a_2, \cdots, a_n \in B$ and $B_A \cong A_A$. Since R is weakly R^n-injective and $a_i \in E(R_R)$ for every $1 \leq i \leq n$, then there exists $X_R \subseteq R_R$ such that $a_i \in X \cong R_R$ for every i, choose $B = X \cap E(A_A)$. Therefore, $a_i \in B$ for every i and $B_A = X \cap A_A \cong R_R \cap E(A_A) = A_A$, hence A is weakly injective. □

Theorem 4.2. *Let R be a radical over subring A and R is weakly R-injective, then R is weakly A-injective.*

Proof. Let $Q = E(R_R)$ and $S = E(A_A)$. Suppose that R is weakly R-injective ring, we will show that A is weakly A-injective. Let $s \in S$ be an arbitrary element, we claim that $sA \subseteq X \cong A_A$. It is clear that $sA \subseteq sR$, where sR

is a cyclic R-submodule of Q. Since is weakly R-injective ring, there exists $Y \subseteq Q$ such that $sA \subseteq sR \subseteq Y \cong R_R$. We choose $X = Y \cap S \subseteq S$. Thus $sA \subseteq Y \cap S = X$. Since $Y \cong R_R$, then $X = Y \cap S \cong R_R \cap S = A_A$. Hence A is weakly A-injective ring. □

Acknowledgements

The author wish to express their since thanks to Prof. M. H. Fahmy for his careful reading, helpful comments and valuable remarks which improved the results of this paper.

References

[1] A. W. Chatter and C. R. Hajarnavis; *Rings with chains conditions*, Research Notes in Mathematics, Pitman 44, London, 1980.

[2] C. Faith; *Radical extensions of rings*, Proc. Amer. Math. Soc., 12 (1961), 274-283.

[3] B. Fe;zenszwalp; *Rings radical over subrings*, Israel, J. Math. 23(2) (1976), 156-164.

[4] I. N. Herstein; *On the hypercenter of ring*, J. Algebra, 36 (1975), 151-157.

[5] N. jacobson; *Structure of Rings*, Americam Mathematical Socity (1964).

[6] S. K. Jain and S. R. Lopez-Permouth; *Rings whose cyclic are essentially embeddable in projective module*, J. Algebra 128 (1990), 257-269.

[7] S. Lange, Algebra, Adison-Wsley, Mass., 1965.

[8] B. Lin; *H-Extension of Rings*, J. Austr. Math. Soc., 10, No.1-2(1969), 236-241 .

[9] Ya. N. Mahsev; *H-Extension of Rings with Identity Relations*, Algebra i logika, Vol. 10, No. 5, (1971), 495-504.

[10] C. Procesi; *Non-commutative Jacobson ring*, Annali dela Scuola Norm. Sup. Pisa Ser. III 21 (1967), 281-289.

Received: January, 2012