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Ricci Flow as a Gradient Flow on Some Quasi Einstein Manifolds

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Abstract

In this paper we study gradient Ricci flow on n dimensional quasi Einstein manifold with an example on 5-dimension. We have also studied quasi conformally flat quasi Einstein manifold and gradient Ricci flow on four dimensional quasi conformally flat quasi Einstein manifold.

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1 Introduction.

We start with a smooth closed (that is, compact and without boundary) manifold M, equiped with a smooth Riemannian metric g. Ricci flow is a means of processing the metric g by allowing it to evolve under the PDE

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1.1}$$

where R_{ij} is Ricci curvature tensor which depends upon g_{ij} [9].

Ricci flow was introduced by R.S.Hamilton in 1982 and after him he himself [11], [12], [13] and many authors such as Perelman [4], [5], S.T.Yau, B.Chow, Morgan [6] and others, worked on it.

In simple situation, the flow can be used to deform g into a metric distinguished by its curvature. The behaviour of the flow serve to tell us much about the topology of the underlying manifold. A simple example of a Ricci flow is that starting from a round sphere, the flow will shrink homothetically to a point in finite time.

More generally, if we consider a metric g such that $R_{ij}(g_0) = \lambda\, g_0$ for some constant $\lambda \in R$, then a solution of $g(t)$ of (1.1) with $g(0) = g_0$ is given by

$$g(t) = (1 - 2\lambda t)g_0 \tag{1.2}$$

In particular, for the round 'unit' sphere (S^n, g_0) we have
$Ric(g_0) = (n-1)g_0$. So the evolution is

$$g(t) = (1 - 2(n-1)t)g_0$$

and the sphere collapses to a point at time $T = \frac{1}{2(n-1)}$.

On the other hand , if we take g_0 as a hyperbolic metric i.e., of constant sectional curvature -1 then

$$Ric(g_0) = -(n-1)g_0$$

and the evolution is

$$g(t) = (1 + 2(n-1)t)g_0$$

and the manifold expands homothetically for all time.
Our main objectives are to study the nature of gradient Ricci flow on n dimensional quasi Einstein manifold, quasi conformally flat quasi Einstein manifold $(QE)_n$.
This gradient flow is just the Ricci flow,modified by a time-dependent diffeomorphism.That is,there exists a smooth family of diffeomorphisms $\phi_t : M \longrightarrow M$ such that ${\phi_t}^*(g(t))$ is the Ricci flow starting at g_0 [9].
In 1968 Yano and Sawaki [7] introduced the notion of a new curvature tensor called quasi conformal curvature tensor which includes both the conformal and concircular tensor as special cases.
The quasi-conformal curvature tensor $\hat{C}^h_{ijk}$ of type (1,3) of a Riemannian space of dimension $(n \geq 3)$ is defined by

$$\hat{C}^h_{ijk} = -(n-2)bC^h_{ijk} + [a + (n-2)b]\tilde{C}^h_{ijk} \tag{1.3}$$

where a, b are arbitrary constant not simultaneously zero, C^h_{ijk} and $\tilde{C}^h_{ijk}$ are conformal and concircular curvature tensor of type (1,3) respectively are given by

$$\begin{aligned} C^h_{ijk} &= R^h_{ijk} - \tfrac{1}{(n-2)}[\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}] \\ &+ \tfrac{r}{(n-1)(n-2)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}] \end{aligned} \tag{1.4}$$

and $$\tilde{C}^h_{ijk} = R^h_{ijk} - \tfrac{r}{n(n-1)}[\delta^h_k g_{ij} - \delta^h_j g_{ik}] \tag{1.5}$$

where r denotes the scalar curvature of the space. Using (1.4) and (1.5) in (1.3) we get

$$\begin{aligned} C'^h_{ijk} &= aR^h_{ijk} + b[\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}] \\ &- \tfrac{r}{n}(\tfrac{a}{n-1} + 2b)[\delta^h_k g_{ij} - \delta^h_j g_{ik}] \end{aligned} \tag{1.6}$$

In global form conformal curtvature tensor can be written as

$$\begin{aligned} C'(X,Y)Z &= aR(X,Y)Z - b\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y\} + \tfrac{r}{n}(\tfrac{a}{n-1} + 2b)\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{1.7}$$

Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. A Riemannian space of dimension $(n > 3)$ is said to be quasi-conformally flat if its quasi-conformal curvature tensor vanishes identically.
Again a non-flat Riemannian manifold (M^n, g_0), $(n \geq 3)$ is said to be a quasi Einstein manifold denoted by $(QE)_n$ [3] if its Ricci tensor $Ric(g)$ is not identically zero and satisfies the condition

$$Ric(g) = ag + b\omega \otimes \omega, \quad b \neq 0 \tag{1.8}$$

In local form $Ric(g)$ is denoted by R_{ij}. $\omega = g(., \rho)$ is a non zero 1-form, ρ being a unit vector field and a, b are scalars called associated scalars, ω is called assiciated 1-form and ρ is called the generator of the manifold.
We consider a $(QE)_n$ with associated scalars a, b; associated 1-form A and generator U. Since U is a unit vector field

$$g(U,U) = 1 \tag{1.9}$$

Also $$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.10}$$

Contracing (1.9) over X and Y we get

$$r = na + b \tag{1.11}$$

Let (M^n, g) be a quasi Einstein Manifold.If the generator ξ belongs to the $k-$nullity distribution $N(k)$ [8] for some smooth function k,then we say that (M^n, g) is an $N(k)-$ quasi Einstein manifold.
We know from [10] that the Ricci tensor of a 3-dimensional pseudo symmetric semi Riemannian manifold satisfies (1.8) and hence a 3-dimensional pseudo symmetric semi Riemannian manifold in the sense of Deszez is a quasi Einstein manifold. In [2] authors have studied Ricci flow on quasi Einstein manifold. M.M.Tripathi an Jeong-Sik Kim studied $N(k)-$quasi Einstein manifold in [8]. In this paper we obtain solution of gradient Ricci flow on quasi Einstein manifold of dimension ≥ 3. Later we show that an $n-$dimensional quasi conformally flat quasi Einstein manifold is an $N(k)$ quasi Einstein manifold along with an example. At last we study gradient Ricci flow on 4-dimensional quasi conformally flat quasi Einstein manifold and find solutions of a considered metric.

2. Gradient Ricci flow on $n, (n \geq 3)$ dimensional quasi Einstein manifold.

Given a compact manifold with a Riemannian metric g, where the total scalar curvature is denoted by $E(g) = \int_M r dv$.

et us consider the first variation of E under an arbitrary change of metric. Generally we write $h = \frac{\partial g}{\partial t}$ and by the two conditions

$$\pounds(df) \neq g = \pounds(\nabla f)g = 2Hess(g) \tag{2.0}$$

and
$$trh(R(X,.)W,.) =< Rm(X,.,W,.), h > \tag{2.1}$$

by computing we get from [9],

$$\begin{aligned}\frac{d}{dt}\int r dv &= \int \frac{\partial r}{\partial t} dv + \int r.\frac{1}{2}(trh)dv \\ &= \int - < Ric, h > +\delta^2 h - \Delta(trh)dv + \int \frac{r}{2}(trh)dv \\ &= \int < \frac{r}{2}g - Ric, h > dv\end{aligned}$$

hence the gradient of E is then given by

$$\nabla E(g) = \tfrac{r}{2}g - Ric \tag{2.2}$$

Considering the relation $R_{ij} = ag_{ij} + bA_iA_j$ which holds for a quasi Einstein manifold (M^n, g) on $U \subset M$ where,

$$U = \{x : R_{ij} \neq \tfrac{r}{n}g_{ij}\ at\ x\}$$

and A_i is a unit covariant vector on U and a, b are some scalars on U, we get from(2.2)

$$\nabla E(g) = \tfrac{r}{2}g - (ag + bA \otimes A)$$

Hence

$$\tfrac{\partial g_{ij}}{\partial t} = \tfrac{r}{2}g_{ij} - ag_{ij} - bA_iA_j$$

i.e. $$\tfrac{\partial g_{ij}}{\partial t} = (\tfrac{na+b}{2} - a)g_{ij} - bA_iA_j$$

By integrating we get

$$g_{ij} = \{(\tfrac{na+b}{2} - a)g_0 + b\acute{g}_0\ddot{g}_0\}t + c$$

Where c is an arbitrary constant and $\acute{g}_0 = g(X, \rho)$,$\ddot{g}_0 = g(Y, \rho)$. At $t = 0$, $g(0) = g_0$ imply $c = g_0$, hence

$$g(t) = \{(\tfrac{na+b}{2} - a)t + 1)\}g_0 + bt\acute{g}_0\ddot{g}_0\}$$

Hence we can state the theorem as,

Theorem. *If (M^n, g) be a quasi Einstein manifold $n \geq 3$ then the solution of gradient Ricci flow will be*

$$g(t) = \{(\tfrac{na+b}{2} - a)t + 1)\}g_0 + bt\acute{g}_0\ddot{g}_0$$

Now we may consider the flow of metrics follows from (2.2) as

$$\tfrac{\partial g}{\partial t} = \tfrac{r}{2}g - Ric$$

i.e, $$\tfrac{\partial g_{ij}}{\partial t} = \tfrac{r}{2}g_{ij} - R_{ij} \tag{2.3}$$

By the proposition obtained from [9],we have
$\frac{\partial r}{\partial t} = -\langle Ric, h\rangle + \delta^2 h - \Delta trh$

the identity $\delta Ric + \frac{1}{2}dr = 0$ reduces to

$$\frac{\partial r}{\partial t} = -(\tfrac{n}{2}-1)\delta r + |Ric|^2 - \tfrac{1}{2}r^2 \tag{2.4}$$

Which is a backward heat equation.
Example: Here we consider a symmetric tensor field in $(QE)_5$ defined by [14]

$$g_{ij} = \begin{pmatrix} \frac{1}{4}(1+(y^1)^2) & \frac{y^1y^2}{4} & 0 & 0 & -\frac{1}{4}y^1 \\ \frac{1}{4}y^1y^2 & \frac{1}{4}(1+(y^2)^2) & 0 & 0 & -\frac{1}{4}y^2 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ -\frac{1}{4}y^1 & -\frac{1}{4}y^2 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{2.5}$$

and $\quad R_{ij} = -2g_{ij} + bA_iA_j$

After some brief calculations it follows that the Ricci tensor has the following non zero components

$$R_{\delta\alpha} = -\tfrac{1}{2}(\delta_{\alpha\delta} - 2y^\alpha y^\delta)$$

$$R_{\delta^*\alpha} = 0, R_{\Delta\Delta} = 1$$

we have R_{ij},where $i,j = 1,2,...5$ and y^1, y^2 are standard coordinates of M^5.

Here we consider $r = na + b$ where $n = 5, a = -2, b = 1$. Hence the partial differential equations of the gradient Ricci flow from the matrix g_{ij} of $(QE)_5$ are given by

(i) $\quad \frac{\partial g_{11}}{\partial t} = \frac{r}{2}g_{11} - R_{11}$

and the solution becomes

$\frac{1}{34}\log(5 + 17(y^1)^2) = -\frac{1}{4}t + c$

(ii) $\quad \frac{\partial g_{12}}{\partial t} = \frac{r}{2}g_{12} - R_{12}$

where $g_{12} = \frac{y^1y^2}{4}$

and $R_{12} = -\frac{1}{2}(\delta_{21} - 2y^1y^2) = -\frac{1}{2}(-2y^1y^2) = y^1y^2$

Hence $\quad \frac{\partial}{\partial t}\frac{1}{4}(y^1y^2) = -\frac{9}{2}\frac{y^1y^2}{4} - y^1y^2$

and the solution is

$$-\frac{2}{17}\log(y^1y^2) = c(const)$$

(iii) $$\frac{\partial g_{13}}{\partial t} = \frac{r}{2}g_{13} - R_{13} = 0$$

(iv) $$\frac{\partial g_{14}}{\partial t} = 0$$

similarly we can get the other solutions for g_{ij}.

Next from (2.4)

$$\frac{\partial r}{\partial t} = -(\frac{n}{2} - 1)\Delta r + |Ric|^2 - \frac{1}{2}r^2$$

where $\quad n = 5, r = -9$

hence $\quad \frac{\partial(-9)}{\partial t} = -(\frac{5}{2} - 1)\Delta(-9) + |R_{ij}|^2 - \frac{1}{2}(-9)^2$

or, $\quad 0 = 0 + (R_{ij})^2 - \frac{1}{2}(9)^2$

or, $\quad R_{ij} = \pm\frac{9}{\sqrt{2}}$

3. Quasi conformally flat $(QE)_n$ and gradient Ricci flow on $4-$dimensional quasi conformally flat $(QE)_n$.

First we consider quasi conformally flat quasi Einstein manifold, so from (1.7) we get

$$aR(X,Y)Z = -b\{g(Y,Z)QX - g(X,Z)QY + s(Y,Z)X - s(X,Z)Y\} \\ + \frac{R}{n}[\frac{a}{n-1} + 2b]\{g(Y,Z)X - g(X,Z)Y\}$$

Putting $Z = \xi$, we obtain

$$R(X,Y)\xi = -\frac{b}{a}\{g(Y,\xi)(a+b)X - g(X,\xi)(a+b)Y + S(Y,\xi)X - S(X,\xi)Y\}$$

$$+ \frac{r}{an}[\frac{a}{n-1} + 2b]\{g(Y,\xi)X - g(X,\xi)Y\}$$

After some brief calculations we get

$$R(X,Y)\xi = (1 + \frac{b}{an})(\frac{a+2bn-2b}{n-1})\{\eta(Y)X - \eta(X)Y\} \tag{3.1}$$

Result:-*Hence we can state that an n-dimensional quasi conformally flat quasi Einstein manifold is an $N(k)-$ quasi Einstein manifold , where*

$$k = \{(1 + \frac{b}{an})(\frac{a-2b+2bn}{n-1})\}.$$

For example a quasi Einstein manifold with $a = 2,\ b = -1$ and $n = 4$, becomes $N(-\frac{7}{3})$ quasi Einstein manifold.

Now we study gradient Ricci flow on four dimensional quasi conformally flat quasi Einstein manifold. For quasi conformally flat quasi Einstein manifold on the four dimensional real number space $\Re^4$ [1] by the given metric

$$ds^2 = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \tag{3.2}$$

where $0 < x^4 < \infty,\ (x^1, ..., x^4)$ are the standard co-ordinates of $\Re^4$

Here $\quad r = \frac{4}{3}(x^4)^{-2}$

$$R_{11} = R_{22} = R_{33} = \frac{2}{3}(x^4)^{-\frac{2}{3}}, R_{44} = -\frac{2}{3}(x^4)^{-2}$$

We have to find the gradient Ricci flow on this manifold.

From (2.2) we know $\quad \frac{\partial g_{ij}}{\partial t} = \frac{r}{2}g_{ij} - R_{ij}$

so for $n = 4,\ r = \frac{4}{3}(x^4)^{-2}$ and $g_{11} = g_{22} = g_{33} = (x^4)^{\frac{4}{3}}$

we get

$$\frac{\partial}{\partial t}(x^4)^{\frac{4}{3}} = \frac{2}{3}(x^4)^{-\frac{2}{3}} - \frac{2}{3}(x^4)^{-\frac{2}{3}} = 0$$

i.e, $\quad x^4 = const$

and for $g_{44} = 1,$ we have

$$(x^4)^{-2} = 0$$

Hence for a four dimensional quasi conformally flat quasi Einstein manifold with the metric (3.2), the solutions of gradient Ricci flow are as above.

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