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Normalizer of Sylow Subgroups and the Structure of a Finite Group[1]

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Abstract

In this paper, we prove the p-nilpotency of a finite group with assumption that some subgroups of Sylow subgroup are weakly s-semipermutable subgroups in the normalizer of Sylow subgroups. Our results unify and generalize some earlier results.

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1 Introduction

In this paper, all groups considered are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group G. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ the class of supersolvable groups. The other notation and terminology are standard(see [5]).

For two subgroups H and K of G, we say H permutes with K if $HK = KH$. We say, following Chen [3], A subgroup H of a group G is said to

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be s-semipermutable, or s-seminormal in G if it permutes with all Sylow p-subgroups P of G with $(p, |H|) = 1$. Recently, Xu and Li [8] introduced a new embedding property, namely, the weakly s-semipermutability of subgroups of a group. A subgroup H of a group G is said to be weakly s-semipermutable in G if G has a subnormal subgroup K such that $HK = G$ and $H \cap K \leq H_{\bar{s}G}$, where $H_{\bar{s}G}$ is the subgroup of H generated by all subgroups of H which are s-semipermutable in G. The authors presented some conditions for a group to be p-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup are weakly s-semipermutable subgroups (see [9]).

On the other hand, normalizer of Sylow subgroups of a group play an important role in the structure of a group. Let P be a Sylow subgroup of a group G. A question who is always interesting in is the relation between the property of the normalizer of P and property of G. Many results have been obtained. For example, well know Burnside′s Theorem.

Theorem 1.1. *(Burnside) Let P be a Sylow p-subgroup of G. If $N_G(P) = C_G(P)$, then G is p-nilpotent.*

Then, Hall in [6] got the following generalization of Burnside′s Theorem:

Theorem 1.2. *Let P be a Sylow p-subgroup of G. If p'-elements of $N_G(P)$ are commute to the elements of P and the class size of P is less than p, then G is p-nilpotent.*

Wielandt, Ballester-Bolinches and Esteban-Romero proved the following respectively.

Theorem 1.3. *([5]) A group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

Theorem 1.4. *([2]) A group G is p-nilpotent if it has a modular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

Now, under the assumption that all maximal subgroups of a Sylow subgroup P are weakly s-semipermutable subgroups of $N_G(P)$, we shall establish the structure of a group G. Some interesting new results are obtained.

2 Preliminary results

A subgroup H of a group G is said to be s-permutable, s-quasinormal, or π-quasinormal in G if $PH = HP$ for all Sylow subgroups P of G (see [7]).

Lemma 2.1. *([7] and [10, Lemma 2.2]) Suppose that U is π-quasinormal in a group G, $H \leq G$ and K a normal subgroup of G. Then:*

(a) If $U \leq H$, then U is π-quasinormal in H;

(b) UK is π-quasinormal in G and UK/K is π-quasinormal in G/K;

(c) Let $K \leq H$ and H/K is π-quasinormal in G/K, then H is π-quasinormal in G;

(d) If P is π-quasinormal subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.

Lemma 2.2. *([8, Lemma 2.3]) Let G be a group and $A \leq E \leq G$. Then:*

(1) Let $N \trianglelefteq G$, $N \leq A$ and A is a p-group. If A is weakly s-semipermutable in G, then A/N is weakly s-semipermutable in G/N.

(2) Suppose that K is normal in G, and A is a p-group, $(|K|, p) = 1$. If A is weakly s-semipermutable in G, then AK/K is weakly s-semipermutable in G/K.

(3) If A is weakly s-semipermutable in G, then A is weakly s-semipermutable in E.

Lemma 2.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a soluble normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every Sylow subgroup of $F(N)$ is cyclic, then $G \in \mathcal{F}$.*

Proof Assume that the result is false and let G be a minimal counterexample on $|G| + |N|$. Suppose that all Sylow subgroups of $F(N)$ are cyclic. Since $F(N)/\Phi(N) = F(N/\Phi(N))$, all Sylow subgroups of $F(N/\Phi(N))$ are cyclic. So $(G/\Phi(N), N/\Phi(N))$ satisfies the hypothesis of Lemma 2.3. Thus if $\Phi(N) \neq 1$, then $G \in \mathcal{F}$. Hence we may assume that $\Phi(N) = 1$. Since $\Phi(F(N)) \subseteq \Phi(N)$, $\Phi(F(N)) = 1$. Let $\pi(F(N)) = \{p_i \mid 1 \leq i \leq t\}$ and $K_i \in Syl_{p_i}(F(N))$, where $p_1 < p_2 < \cdots < p_t$, then $K_i \lhd G$ and $|K_i| = p_i$. So $|Aut(K_i)| = p_i - 1$ and $G/C_G(K_i)$ is isomorphic to a subgroup of $Aut(K_i)$. By $Aut(K_i)$ is a cyclic, we get $G/C_G(K_i)$ is cyclic. Let $U = \cap_{i=1}^{t} C_G(K_i)$, then $G/U \in \mathcal{U}$ and so $G/U \cap N \in \mathcal{F}$. It is easy to see that $F(U \cap N) = F(N)$. Hence if $U \cap N < N$, then $G \in \mathcal{F}$, a contradiction. So we may assume that $N \leq U$. It is clear that $U = C_G(F(N))$. So $N \leq C_G(F(N))$, $F(N) \leq Z(N)$. By N is solvable, $N = C_N(F(N)) \leq F(N)$, so $N = F(N)$ and N is cyclic. Since $(G/K_1)/(N/K_1) \cong G/N \in \mathcal{F}$, and $F(N/K_1) = N/K_1$ is cyclic, by the minimality of $|G| + |N|$, we get $G/K_1 \in \mathcal{F}$. So we may assume that $N = K_1$ and $K_1 \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = K_1 M$, then

$M \cap K_1 = 1$. Since K_1 centralizes $C_G(K_1) \cap M$ and M normalizes $C_G(K_1) \cap M$, we get $C_G(K_1) \cap M \lhd G$. Let $T = C_G(K_1) \cap M$, then $K_1 \nleq T$ and so $K_1 \cap T = 1$. By $G = C_G(K_1)M$, we have $M/T \cong C_G(K_1)M/C_G(K_1) = G/C_G(K_1)$. Since $G/C_G(K_1)$ is cyclic and $G/T = K_1T/T \rtimes M/T$, we have $G/T \in \mathcal{U}$. Thus by $\mathcal{F}$ is a formation, we have obtained that $G \cong G/K_1 \cap T \in \mathcal{F}$. □

Lemma 2.4. *([9, Theorem 3.1]) Suppose that N is a normal subgroup of a group G such that G/N is p-nilpotent and P is a Sylow p-subgroup of N, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If all maximal subgroup of P are weakly s-semipermutable in G, then G is p-nilpotent.*

3 Main Results

Theorem 3.1. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-semipermutable in $N_G(P)$ and $P^{'}$ is π-quasinormal in G, then G is p-nilpotent.*

Proof Assume that the result is false. Let G be a counterexample of minimal order.

(1) Let L be a normal subgroup of G contained in P, then G/L satisfies the hypothesis.

It is clear that $(|G/L|, p-1) = 1$. And for any maximal subgroup P_1/L of P/L, $p = |P/L : P_1/L| = |P : P_1|$, so P_1 is a maximal subgroup of P. By the hypothesis, P_1 is a weakly s-semipermutable in $N_G(P)$ and $P^{'}$ is π-quasinormal in G. By Lemma 2.2 (1), P_1/L is a weakly s-semipermutable in $N_G(P)/L = N_{G/L}(P/L)$, and $(P/L)^{'} = P^{'}L/L$ is π-quasinormal in G/L by Lemma 2.1 (b), thus we have (1).

(2) $1 \neq P^{'} \leq O_p(G)$ and G is solvable.

For any $Q \in Syl_q(N_G(P))$ with $q \neq p$, it is clear that all maximal subgroups of P are weakly s-semipermutable in PQ by Lemma 2.2 (3). Thus PQ satisfy the hypothesis of Lemma 2.4, so PQ is p-nilpotent, hence $Q \leq C_G(P)$. Assume that P is abelian, when q run over $\pi(N_G(P))$, we have $N_G(P) = C_G(P)$, hence G is p-nilpotent by Burnside's Theorem, a contradiction. So $P^{'} \neq 1$ and $P^{'}$ is π-quasinormal in G, thus $P^{'} \lhd \lhd G$, therefore, $O_p(G) \neq 1$. By (1), we have $G/O_p(G)$ is p-nilpotent. Since $(|G|, p-1) = 1$, we conclude that $G/O_p(G)$ is solvable, thus G is solvable.

(3) End of the proof.

Let $\{G_r \mid r \in \pi(G)\}$ be a Sylow system of G and $H = G_pG_r$ for any $r \in \pi(G)$ with $r \neq p$. By Lemma 2.1(a) and Lemma 2.2 (3), the hypothesis is still true for H. If $|\pi(G)| > 2$, then $G_r \trianglelefteq H$, which implies that G_p normalizes

G_r for any $r \in \pi(G)$, hence G is p-nilpotent, a contradiction. Thus we may assume that $|G| = p^a q^b$.

Let L be a minimal normal subgroup of G. Since P' is π-qusinormal in G and $L \trianglelefteq G$, $P'L$ is π-qusinormal in G. Thus $P'L/L$ is π-qusinormal in G/L by Lemma 2.1. If L is a q-group, then we consider the quotient group G/L. Evidently, $PL/L \in Syl_p(G/L)$. For any maximal subgroup T/L of PL/L, we have $p = |(PL/L) : (T/L)|$, and

$$T = PL \cap T = (P \cap T)L.$$

Let $P_1 = P \cap T$, then $P_1 \cap L = P \cap T \cap L = P \cap L$, so

$$p = |PL : T| = |PL : (P \cap T)L| = |P : P \cap T| = |P : P_1|.$$

Thus P_1 is a maximal subgroup of P. By the hypothesis, P_1 is a weakly s-semipermutable in $N_G(P)$. Then there exists a subnormal subgroup K of $N_G(P)$ such that $N_G(P) = P_1K$ and $P_1 \cap K \leq (P_1)_{\overline{s}N_G(P)}$. Obviously, we have $N_G(P)L/L = (P_1L/L)(KL/L)$ and $KL/L \trianglelefteq \trianglelefteq N_G(P)L/L$. Since $(|P_1|, |L|) = 1$, we get

$$|P_1 \cap KL| = \frac{|P_1| \cdot |KL|_p}{|P_1KL|_p} = \frac{|P_1| \cdot |K|_p}{|N_G(P)L|_p} = \frac{|P_1| \cdot |K|_p}{|N_G(P)|_p} = |P_1 \cap K|.$$

This implies $P_1 \cap KL = P_1 \cap K$, then

$$(P_1L/L) \cap (KL/L) = (P_1L \cap KL)/L = (P_1 \cap KL)L/L$$

$$= (P_1 \cap K)L/L \leq (P_1)_{\overline{s}N_G(P)}L/L \leq (P_1L/L)_{\overline{s}N_{G/L}(PL/L)}.$$

thus G/L is p-nilpotent by the minimality of G. Hence G is p-nilpotent, a contradiction. Therefore, L is a p-group and $L \leq P$. By (1), G/L is p-nilpotent. Similarly, if N is another minimal normal subgroup of G, then $N \leq P$ and so G/N is also p-nilpotent. Now it follows that $G \cong G/N \cap L$ is p-nilpotent, a contradiction. Thus L must be the unique minimal normal subgroup of G. Since the class of p-nilpotent group is saturated formation, $L \nleq \Phi(G)$, hence $\Phi(G) = 1$. By [11, Thorem 5.3], we get that $O_p(G) = F(G) = L$.

Since $P^{'}$ is π-quasinormal in G, we have $N_G(P^{'}) \geq O^p(G)$ by Lemma 2.1 (d), and P normalizes $P^{'}$, we get $P^{'} \trianglelefteq G$, then $P^{'} = O_p(G) = L$ by the unique minimal normality of L. By (1), we have $G/O_p(G)$ is p-nilpotent, hence $O_p(G)Q \trianglelefteq G$, where $Q \in Syl_q(G)$. Since $O_p(G)Q \cap P = O_p(G) = P^{'} \leq \Phi(P)$, $O_p(G)Q$ is p-nilpotent by J. Tate Theorem([5, Theorem 4.4.7]). Thus $Q \trianglelefteq O_p(G)Q \trianglelefteq G$, this implies $Q \trianglelefteq G$, G is p-nilpotent, a contradiction. This contradiction completes the proof of this theorem. □

Corollary 3.1. *([4, Theorem 3.1]) Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is π-quasinormal in $N_G(P)$ and P' is π-quasinormal in G, then G is p-nilpotent.*

Theorem 3.2. *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is weakly s-semipermutable in $N_G(P)$ and P' is π-quasinormal in G, then G is p-nilpotent.*

Proof Assume that the result is false. Let G be a minimal counterexample with least $|G| + |H|$.

By Lemma 2.1 (a) and Lemma 2.2 (3), it is clear that every maximal subgroup of P is weakly s-semipermutable in $N_H(P)$ and P' is π-quasinormal in H, then H is p-nilpotent by Theorem 3.1. Let M be a normal p-complement of H, then $M \trianglelefteq G$. Assume that $M \neq 1$. We consider the quotient subgroup G/M. Similar to the proof of (3) in Theorem 3.1, it is easy to see that the hypothesis is still true for $(G/M, H/M)$, hence G/M is p-nilpotent, and so G is p-nilpotent, a contradiction. Thus we conclude that $M = 1$. Now $H = P$ is a p-subgroup. Let T/P be a normal p-complement of G/P, it is clear that every maximal subgroup of P is weakly s-semipermutable in $N_T(P)$ and P' is π-quasinormal in T, then T is p-nilpotent by Theorem 3.1, so $T_{p'} \trianglelefteq T \trianglelefteq G$ and $T_{p'}$ is also a Hall p'-subgroup of G, thus $T_{p'} \trianglelefteq G$, hence G is p-nilpotent, a contradiction. This contradiction completes the proof of this theorem. □

Theorem 3.3. *Let G be a group. For any prime factor p of $|G|$, there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-semipermutable in $N_G(P)$ and P' is π-quasinormal in G, then G is supersolvable.*

Proof Assume that the result is false. Let G be a counterexample of minimal order.

By Theorem 3.1, we know that G is a Sylow tower group of supersolvable type, so G is soluble. Let L be a minimal normal subgroup of G, then L is an elementary r-subgroup, where $r \in \pi(G)$, by Lemma 2.2 (1) and 2.2 (2), we have G/L satisfies the hypothesis, thus G/L is supersolvable by the minimal choice of G. Since the class of supersolvable subgroup is a saturated formation, we may assume that L is the unique minimal normal subgroup of G and $L \nleq \Phi(G)$, so there exists a maximal subgroup M of G such that $G = LM$ with $L \cap M = 1$. Let $q = max\pi(G)$ and $Q \in Syl_q(G)$, then $Q \trianglelefteq G$, thus $L \leq Q$ by the unique minimal normality of L in G. Since $Q = O_q(G) \leq F(G) \leq C_G(L)$, L and M

normalize $Q\cap M$, we get $Q\cap M \lhd G$. So $Q\cap M = 1$ or $L \leq Q\cap M$. If the later happened, then $L \leq M$, that is, $G = LM = M$, a contradiction. So $Q\cap M = 1$, and $L \cap M = 1$. This implies that $|Q| = |G : W| = |L|$, hence $L = Q$. Let Q_1 be a maximal subgroup of $Q = L$. By the hypothesis, Q_1 is weakly s-semipermutable in $G = N_G(Q)$, then there exists a subnormal subgroup K of G such that $G = Q_1K$ and $Q_1 \cap K \leq (Q_1)_{\overline{s}G}$. Clearly, $(Q_1)_{\overline{s}G}Q = Q(Q_1)_{\overline{s}G}$, then $(Q_1)_{\overline{s}G}$ is π-quasinormal in G, so $N_G((Q_1)_{\overline{s}G}) \geq O^q(G)$ by Lemma 2.1 (d). Since Q is an elementary r-subgroup, we get $(Q_1)_{\overline{s}G} \trianglelefteq G$. By the unique minimal normality of L in G and $L = Q$, we have $(Q_1)_{\overline{s}G} = 1$, so $Q_1 \cap K = 1$. On the other hand, since $Q \cap K \lhd \langle Q, K\rangle = G$, we have $Q \cap K = L = Q$ by the unique minimal normality of L in G, so $Q \leq K$. Thus $K = G$. Since $Q_1 \leq K$ and $Q_1 \cap K = 1$, we get $Q_1 = 1$ namely, $|L| = |Q| = q$. By Lemma 2.3, we have G is supersolvable, a contradiction. This contradiction completes the proof of this theorem. □

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup H such that $G/H \in \mathcal{F}$. For any prime factor p of $|H|$, there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is weakly s-semipermutable in $N_G(P)$ and P' is π-quasinormal in G, then $G \in \mathcal{F}$.*

Proof By Lemma 2.2 (3), we have H satisfies the hypothesis of Theorem 3.3, hence H is supersoluble. Let $p = max\pi(H)$ and $P \in Syl_p(H)$, then $P \trianglelefteq G$. We consider the quotient group G/P, then $G/H \cong (G/P)/(H/P) \in \mathcal{F}$. By Lemma 2.1 ($a$) and Lemma 2.2 (1), we have $(G/P, H/P)$ satisfies the hypothesis, thus $G/P \in \mathcal{F}$ by induction. Hence we may assume that $H = P$. By $G = N_G(P)$ and [1, Theorem 1.3], we get $G \in \mathcal{F}$. □

Corollary 3.2. *([4, Theorem 3.4]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup H such that $G/H \in \mathcal{F}$. For any prime factor p of $|H|$, there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is π-quasinormal in $N_G(P)$ and P' is π-quasinormal in G, then $G \in \mathcal{F}$.*

Recall that a group G was called an $\mathcal{A}$-group if all of its Sylow subgroups are abelian. Let G be an $\mathcal{A}$-group, then for any $P \in Syl_p(G)$, we have $P' = 1$, of course, it is π-quasinormal in G, so we have the following corollary.

Corollary 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with normal $\mathcal{A}$-subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup P of H is weakly s-semipermutable in $N_G(P)$, then $G \in \mathcal{F}$.*

4 Remarks

Remark 4.1. We point out that in Theorem 3.1, the condition, P' is π-quasinormal in G, can not be removed.

Example Let $G = PSL_2(q)$, where $q \equiv 1 \pmod 8$. Let P be a Sylow 2-subgroup of G. By [5, II, Theorem 8.27], we have the Sylow 2-subgroup of $PSL_2(q)$ is selfnormalizing in $PSL_2(q)$. Evidently, every maximal subgroup of P is normal in $N_G(P) = P$, and therefore every maximal subgroup of P is weakly s-semipermutable subgroup of $N_G(P)$. However, G is not 2-nilpotent.

Remark 4.2. Even if G is solvable group and p is an odd prime, the hypothesis in Theorem 3.1 that "P' is π-quasinormal in G" could not still omitted.

Example Let $H = Z_3 \times Z_3 \times Z_3$ be an elementary abelian 3-group. It is clear that $Aut(H)$ has a subgroup $Z_{13} \rtimes Z_3$. Now suppose that

$$G = (Z_3 \times Z_3 \times Z_3) \rtimes (Z_{13} \rtimes Z_3)$$

Let P_3 be a Sylow 3-subgroup of G. It is obvious that $N_G(P_3) = P_3$ and therefore every maximal subgroup of P_3 is weakly s-semipermutable subgroup of $N_G(P_3) = P_3$. However, G is not 3-nilpotent.

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References

[1] M. Asaad, On maximal subgroup of finite group, Comm Algebra, 26(11) (1998), 3647-3652.

[2] A. Ballester-Bolinches and R. Esteban-Romero, Sylow permutable subnormal subgroups of finite groups, Journal of Algebra, 251 (2002), 727-738

[3] Zhongmu Chen, On a theorem of Srinivasan, J. of Southwest Normal Univ. Nat. Sci., 12(1) (1987), 1-4.

[4] Xiuyun Guo, Xianhe Zhao, π-quasinormality of the Maximal Subgroups of a Sylow Subgroup in a Local Subgroup, Acta Mathematica Scientia (in Chinese), 28A(6) (2008), 1222-1226.

[5] B. Huppert, Endliche Gruppen I, Springer, New York, Berlin, 1967

[6] P. Hall, On a theorem of Frobenius, Proc. London Math. Soc., 40 (1936), 468-501

[7] O. Kegel, Sylow-Gruppen and Subnormalteiler endlicher Gruppen, Math Z., 78 (1962), 205-211

[8] Yong Xu, Xianhua Li, On weakly s-semipermutable subgroups of finite groups, Front. Math. China, 6(1) (2011), 161-175.

[9] X. H. Li, Y. Xu and T. Zhao, Weakly s-semipermutable subgroups and p-nilpotentcy of finite groups. to appear.

[10] Y. Li, Y. Wang, H. Wei, Influence of π-quasinormality of some subgroups of a finite group, Arch Math., 81 (2003), 245-252

[11] Mingyao Xu, A Introduction to Finite Group, Science Publisher, Beijing, 1999 (in Chinese)

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