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Grammian Solution of the vcKP Equation

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Abstract

The Grammian solution of the variable-coefficient KP (vcKP) equation is given. Then, a vcKP equation with self-consistent sources is presented by using the source generation procedure. Moreover, by applying the Pfaffianization procedure, a new integrable coupling system of the vcKP equation is established. Furthermore, a novel integrable system of the vcKP equation with self-consistent sources is constructed.

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Keywords: Variable-coefficient KP equation, Self-consistent sources, Grammian solution, Integrable system

1 Introduction

Soliton theory plays a very important role in the study of nonlinear science and has been applied in some natural sciences especially in the physics, such as field theory, fluid dynamics and so on. In recent years, the study of integrable systems of soliton equations has attracted much attention. In addition, searching for exact solutions of nonlinear soliton equations is an active and important topic in the soliton theory. Many methods have been developed, such as the inverse scattering method [1], Darboux transformation [2],the Hirota method [3], the source generation procedure [4] and so on. Very recently, variable-coefficient soliton equations have been receiving growing attention. The vcKP equation is of physical and mathematical importance [5-6]. In particular, soliton equations with self-consistent sources are an important class of integrable equations. Following the pioneering work by Melnikov [7], a number of interesting contributions have been made to the study of soliton equations with self-consistent sources via inverse scattering method, Darboux transformations and Hirota bilinear method [8-11]. The purpose of this paper is to construct variable-coefficient soliton equations with self-consistent sources by using the source generation procedure.

In 1991, Hirota and Ohta developed a procedure for generalizing nonlinear evolution equations from the Kadomtsev-Petviashvili hierarchy to produce coupled systems of equations, which is now called Pfaffianization [12,13]. In this paper, we apply the Pfaffianization procedure to the vcKP equation. We will construct a new coupled vcKP equation.

2 Grammian solution of the vcKP equation

We consider the following variable-coefficient KP equation [6]

$$[u_t + 6\beta(t)uu_x + \beta(t)u_{xxx}]_x + \gamma(t)u_{yy} = 0, \tag{1}$$

where $\beta(t)$ and $\gamma(t)$ are arbitrary functions of the variable t. Through the dependent variable transformation $u = 2(\ln\tau)_{xx}$, Eq.(1) can be transformed into the following bilinear form

$$D_x D_t \tau\cdot\tau + \beta(t)D_x^4\tau\cdot\tau + \gamma(t)D_y^2\tau\cdot\tau = 0, \tag{2}$$

where the Hirota bilinear operator D is defined by [13]

$$D_x^l D_y^m D_t^n a\cdot b = (\partial_x - \partial_{x'})^l(\partial_y - \partial_{y'})^m(\partial_t - \partial_{t'})^n a(x,y,t)b(x',y',t')|_{x'=x,y'=y,t'=t}.$$

Theorem 1. *The bilinear equation (2) has the following Gramm-type determinant solution*

$$\tau = \det|\, c_{ij} + \int^x \phi_i\varphi_j dx \,|_{1\le i,j\le N}, \qquad c_{ij} = const., \tag{3}$$

where ϕ_i and φ_j satisfy the linear differential equations

$$\phi_{iy} = \sqrt{\frac{3\beta(t)}{\gamma(t)}}\phi_{ixx}, \phi_{it} = -4\beta(t)\phi_{ixxx}, \varphi_{jy} = -\sqrt{\frac{3\beta(t)}{\gamma(t)}}\varphi_{jxx}, \varphi_{jt} = -4\beta(t)\varphi_{jxxx}. \tag{4}$$

In the following, we will construct the vcKP equation with self-consistent sources. We set the function Θ in the following form:

$$\Theta = \det(c_{ij}(t) + \int^x \phi_i\varphi_j dx)_{1\le i,j\le N} = (1,2,\cdots,N,N^*,\cdots,2^*,1^*), \tag{5}$$

where

$$c_{ij}(t) \equiv \begin{cases} c_j(t), & i=j \quad \text{and} \quad 1\le i\le K\le N, \\ c_{ij}, & i\ne j \quad \text{and} \quad 1\le i,j\le N, \end{cases}$$

with $c_j(t)$ being an arbitrary function of t and K being a positive integer, and ϕ_i, φ_j still satisfy relations (4). Here the Pfaffian elements are defined by

$(i,j^*) = c_{ij}(t) + \int^x \phi_i\varphi_j dx$, $(i,j)=(i^*,j^*)=0$, $i,j=1,2,\cdots,N$. Through the property of determinants [13] $\det(a_{i,j}-x_iy_j)_{1\le i,j\le N} = \det(a_{i,j})_{1\le i,j\le N} - \sum_{i,j=1}^N x_iy_j\Delta_{i,j}$, where $\Delta_{i,j}$ denotes the algebraic cofactor of $\det(a_{i,j})_{1\le i,j\le N}$, we can calculate Θ as follows:

$$\Theta = (1,2,\cdots,N,N^*,\cdots,2^*,1^*) \equiv (\bullet), \qquad \Theta_x = (d_0,d_0^*,\bullet), \tag{6}$$

$$\Theta_y = \sqrt{\frac{3\beta(t)}{\gamma(t)}}[(d_1,d_0^*,\bullet)-(d_0,d_1^*,\bullet)],\ \Theta_{yx} = \sqrt{\frac{3\beta(t)}{\gamma(t)}}[(d_2,d_0^*,\bullet)-(d_0,d_2^*,\bullet)], \tag{7}$$

$$\Theta_t = \sum_{j=1}^K \dot{c}_j(t)(1,\cdots,\hat{j},\cdots,N,N^*,\cdots,\hat{j}^*,\cdots,1^*) \\ -4\beta(t)[(d_0,d_2^*,\bullet)-(d_1,d_1^*,\bullet)+(d_2,d_0^*,\bullet)], \tag{8}$$

where ˆ indicates deletion of the letter under it, and the dot denotes the derivative of $c_j(t)$ with respect to t. Then, the function Θ will not satisfy equation (2) again, and it just satisfies the following new equation

$$D_xD_t\Theta\cdot\Theta + \beta(t)D_x^4\Theta\cdot\Theta + \gamma(t)D_y^2\Theta\cdot\Theta + \sum_{j=1}^K Q_jP_j, = 0, \tag{9a}$$

$$(\sqrt{\frac{\gamma(t)}{3\beta(t)}}D_y + D_x^2)\Theta\cdot Q_j = 0, \qquad j=1,2,\cdots,K, \tag{9b}$$

$$(\sqrt{\frac{\gamma(t)}{3\beta(t)}}D_y + D_x^2)P_j\cdot\Theta = 0, \qquad j=1,2,\cdots,K, \tag{9c}$$

where Q_j and P_j are given by the following forms:

$$Q_j = \sqrt{2\dot{c}_j(t)}(d_0^*,1,\cdots,N,N^*,\cdots,\hat{j}^*,\cdots,1^*), \quad j=1,2,\cdots,K, \tag{10a}$$

$$P_j = \sqrt{2\dot{c}_j(t)}(d_0,1,\cdots,\hat{j},\cdots,N,N^*,\cdots,1^*), \quad j=1,2,\cdots,K. \tag{10b}$$

In fact, substitution of (6)-(8) and (10) into (9a) leads to the following Pfaffian identities:

$$24\beta(t)[(d_0,d_0^*,d_1,d_1^*,\bullet)(\bullet)-(d_0,d_0^*,\bullet)(d_1,d_1^*,\bullet)+(d_0,d_1^*,\bullet)(d_1,d_0^*,\bullet)] \\ +2\sum_{j=1}^K \dot{c}_j(t)[(d_0,d_0^*,1,\cdots,\hat{j},\cdots,N,N^*,\cdots,\hat{j}^*,\cdots,1^*)(\bullet) \\ +(d_0^*,1,\cdots,N,N^*,\cdots,\hat{j}^*,\cdots,1^*)(d_0,1,\cdots,\hat{j},\cdots,N,N^*,\cdots,1^*) \\ -(d_0,d_0^*,\bullet)(1,\cdots,\hat{j},\cdots,N,N^*,\cdots,\hat{j}^*,\cdots,1^*)] = 0.$$

Therefore, Eq.(9a) holds. In the same way, we can prove that Θ, Q_j and P_j satisfy equations (9b) and (9c).

Theorem 2. *Eqs.(9) are the variable-coefficient KP equation with self-consistent sources. Moreover, Θ, Q_j and P_j in (5) and (10) are their Grammian solutions.*

If we apply the dependent variable transformations $u = 2(\ln\Theta)_{xx}$, $\Phi_j = Q_j/\Theta$, $\Psi_j = P_j/\Theta$, $j = 1, 2, \cdots, K$. Then Eqs.(13) are transformed into the following nonlinear equations:

$$u_t + \beta(t)(u_{xxx} + 6uu_x) + \gamma(t)\int_{-\infty}^{x} u_{yy}dx + \sum_{j=1}^{K}(\Phi_j\Psi_j)_x = 0, \tag{11a}$$

$$\Phi_{j,y} = \Phi_{j,xx} + u\Phi_j, \quad \Psi_{j,y} = -\Psi_{j,xx} - u\Psi_j, \quad j = 1, 2, \cdots, K, \tag{11b}$$

which is just the variable-coefficient KP equation with self-consistent sources.

3 A new coupled system of the vcKP equation

In the following, we will apply the Pfaffianization procedure to the vcKP equation (2). In order to Pfaffianize the vcKP equation (2), we consider the Pfaffian

$$f = (1, 2, \cdots, 2N), \tag{12}$$

whose entries are chosen to satisfy the following differential rules,

$$\frac{\partial}{\partial x}(i,j) = (i+1,j) + (i,j+1), \tag{13a}$$

$$\frac{\partial}{\partial y}(i,j) = \sqrt{\frac{3\beta(t)}{\gamma(t)}}[(i+2,j) + (i,j+2)], \tag{13b}$$

$$\frac{\partial}{\partial t}(i,j) = -4\beta(t)[(i+3,j) + (i,j+3)]. \tag{13c}$$

The function f with entries (13) is called the Wronski-type Pfaffian. As an example, we can take the entry $(i,j) = \sum_{k=1}^{M}[\Phi_k^{(i)}\Psi_k^{(j)} - \Phi_k^{(j)}\Psi_k^{(i)}]$, where M is an arbitrary natural number, $\Phi_k^{(l)}$ and $\Psi_k^{(l)}$ denote the lth derivatives with respect to x, and Φ_k and Ψ_k satisfy

$$\Phi_{ky} = \sqrt{\frac{3\beta(t)}{\gamma(t)}}\Phi_{kxx}, \quad \Phi_{kt} = -4\beta(t)\Phi_{kxxx}, \tag{14a}$$

$$\Psi_{ky} = \sqrt{\frac{3\beta(t)}{\gamma(t)}}\Psi_{kxx}, \quad \Psi_{kt} = -4\beta(t)\Psi_{kxxx}, \tag{14b}$$

Then, we can calculate the differential formulae of f. So, we have

$$\begin{aligned}
&D_xD_tf\cdot f+\beta(t)D_x^4f\cdot f+\gamma(t)D_y^2f\cdot f\\
&\quad=2(f_{xt}f-f_xf_t)+2\beta(t)(f_{xxxx}f-4f_xf_{xxx}+3f_{xx}^2)+2\gamma(t)(f_{yy}f-f_y^2)\\
&\quad=24\beta(t)[(1,2,\cdots,2N)(1,2,\cdots,2N-2,2N+1,2N+2)\\
&\qquad-(1,2,\cdots,2N-1,2N+1)(1,2,\cdots,2N-2,2N,2N+1)\\
&\qquad+(1,2,\cdots,2N-1,2N+2)(1,2,\cdots,2N-2,2N,2N+1)]. \qquad (15)
\end{aligned}$$

These Pfaffians no longer satisfy the bilinear equation (2). According to the Hirota-Ohta procedure, we now introduce two new variables g and h defined by

$$g=(1,2,\cdots,2N-2),\quad h=(1,2,\cdots,2N+2). \qquad (16)$$

Using the Pfaffian identities [13], we can show that f, g and h satisfy the following three bilinear equations

$$D_xD_tf\cdot f+\beta(t)D_x^4f\cdot f+\gamma(t)D_y^2f\cdot f=24\beta(t)gh, \qquad (17a)$$

$$2\beta(t)D_x^3g\cdot f+2\sqrt{3\beta(t)\gamma(t)}D_xD_yg\cdot f-D_tg\cdot f=0, \qquad (17b)$$

$$2\beta(t)D_x^3h\cdot f-2\sqrt{3\beta(t)\gamma(t)}D_xD_yh\cdot f-D_th\cdot f=0. \qquad (17c)$$

Through the dependent variable transformation $u=2(\ln f)_{xx},\ \rho=\frac{g}{f},\ \sigma=\frac{h}{f}$, we can derive the following nonlinear coupled system from (24):

$$u_t+\beta(t)(u_{xxx}+6uu_x)+\gamma(t)\int_{-\infty}^{x}u_{yy}dx-24\beta(t)(\rho\sigma)_x=0, \qquad (18a)$$

$$\rho_t-2\beta(t)(\rho_{xxx}+3u\rho_x)-2\sqrt{3\beta(t)\gamma(t)}(\rho_{xy}+\rho\int_{-\infty}^{x}u_ydx)=0, \qquad (18b)$$

$$\sigma_t-2\beta(t)(\sigma_{xxx}+3u\sigma_x)+2\sqrt{3\beta(t)\gamma(t)}(\sigma_{xy}+\sigma\int_{-\infty}^{x}u_ydx)=0. \qquad (18c)$$

Theorem 3. *The new coupled variable-coefficient KP equation (17) has the Wronski-type Pfaffian solution (12) and (16).*

In the following, we discuss another Pfaffian solution, which is the Gramm-type Pfaffian solution for the coupled vcKP equation (17). The Gramm-type Pfaffian solution is given as follows

$$f=(1,2,\cdots,2N),\ g=(c_1,c_0,1,2,\cdots,2N),\ h=(d_0,d_1,1,2,\cdots,2N). \qquad (19)$$

Each Pfaffian element is defined by

$$(i,j)=c_{ij}+\int^{x}(f_ig_j-f_jg_i)dx,\qquad c_{ij}=-c_{ji}, \qquad (20a)$$

$$(d_n,i)=\frac{\partial^n}{\partial x^n}f_i,\quad (c_n,i)=\frac{\partial^n}{\partial x^n}g_i,\quad (c_m,c_l)=(d_m,d_l)=(d_m,c_l)=0,\tag{20b}$$

where f_i and g_j satisfy the following differential relations

$$f_{i,y}=\sqrt{\frac{3\beta(t)}{\gamma(t)}}f_{i,xx},\ g_{j,y}=-\sqrt{\frac{3\beta(t)}{\gamma(t)}}g_{j,xx},\tag{21a}$$

$$f_{i,t}=-4\beta(t)f_{i,xxx},\ g_{j,t}=-4\beta(t)g_{j,xxx}.\tag{21b}$$

Theorem 4. *The coupled variable-coefficient KP equation (17) has the Gramm-type Pfaffian solution (19).*

4 A novel integrable system

In the following, we will construct a new coupled system of the vcKP equation with self-consistent sources. We assume that the functions f, g and h have the forms:

$$f=(1,2,\cdots,2N)=(*),\qquad g=(c_1,c_0,*),\qquad h=(d_0,d_1,*),\tag{22}$$

where the Pfaffian entries are defined as follows

$$(i,j)=C_{ij}(t)+\int^x(f_ig_j-f_jg_i)dx,\qquad i,j=1,2,\cdots,2N,\tag{23a}$$

$$(d_n,i)=\frac{\partial^n}{\partial x^n}f_i,\quad (c_n,i)=\frac{\partial^n}{\partial x^n}g_i,\quad (c_m,c_l)=(d_m,d_l)=(d_m,c_l)=0,\tag{23b}$$

here f_i and g_j satisfy the relations (21), and with $C_{ij}(t)=-C_{ji}(t)$ satisfying

$$C_{ij}(t)\equiv\begin{cases}C_i(t), & i<j \text{ and } j=2N+1-i,\quad 1\le i\le K\le N,\\ c_{ij}, & i<j \text{ and } j\ne 2N+1-i.\end{cases}$$

Then f, g and h do not satisfy Eqs.(17) any more. They just satisfy the following new equation:

$$D_xD_tf\cdot f+\beta(t)D_x^4f\cdot f+\gamma(t)D_y^2f\cdot f=24\beta(t)gh-\sum_{i=1}^K(G_iH_i-\hat{G}_i\hat{H}_i),\tag{24a}$$

$$2\beta(t)D_x^3g\cdot f+2\sqrt{3\beta(t)\gamma(t)}D_xD_yg\cdot f-D_tg\cdot f=-\frac{1}{2}\sum_{i=1}^K D_x\hat{G}_i\cdot G_i,\tag{24b}$$

$$2\beta(t)D_x^3h\cdot f-2\sqrt{3\beta(t)\gamma(t)}D_xD_yh\cdot f-D_th\cdot f=-\frac{1}{2}\sum_{i=1}^K D_x\hat{H}_i\cdot H_i,\tag{24c}$$

$$(\sqrt{\frac{\gamma(t)}{3\beta(t)}}D_y + D_x^2)G_i \cdot f = 2g\hat{H}_i, \qquad i = 1, 2, \cdots, K, \tag{24d}$$

$$(\sqrt{\frac{\gamma(t)}{3\beta(t)}}D_y + D_x^2)\hat{G}_i \cdot f = 2gH_i, \qquad i = 1, 2, \cdots, K, \tag{24e}$$

$$(\sqrt{\frac{\gamma(t)}{3\beta(t)}}D_y + D_x^2)f \cdot H_i = 2h\hat{G}_i, \qquad i = 1, 2, \cdots, K, \tag{24f}$$

$$(\sqrt{\frac{\gamma(t)}{3\beta(t)}}D_y + D_x^2)f \cdot \hat{H}_i = 2hG_i, \qquad i = 1, 2, \cdots, K, \tag{24g}$$

where, G_i, $\hat{G}_i$, H_i and $\hat{H}_i$ are given by the following forms:

$$G_i = \sqrt{2\dot{C}_i(t)}(c_0, 1, \cdots, \hat{i}, \cdots, 2N), \qquad i = 1, 2, \cdots, K, \tag{25a}$$

$$\hat{G}_i = \sqrt{2\dot{C}_i(t)}(c_0, 1, \cdots, 2N \widehat{+1} - i, \cdots, 2N), \qquad i = 1, 2, \cdots, K, \tag{25b}$$

$$\hat{H}_i = \sqrt{2\dot{C}_i(t)}(d_0, 1, \cdots, \hat{i}, \cdots, 2N), \qquad i = 1, 2, \cdots, K, \tag{25c}$$

$$H_i = \sqrt{2\dot{C}_i(t)}(d_0, 1, \cdots, 2N \widehat{+1} - i, \cdots, 2N), \qquad i = 1, 2, \cdots, K. \tag{25d}$$

Theorem 5. *Eqs.(24) are the new integrable system of the variable-coefficient KP equation with self-consistent sources. Moreover, Eqs.(22) and (25) are their Pfaffian solutions.*

Applying the dependent variable transformation $u = 2(\ln f)_{xx}$, $\rho = \frac{g}{f}$, $\sigma = \frac{h}{f}$, $\xi_i = \frac{G_i}{f}$, $\hat{\xi}_i = \frac{\hat{G}_i}{f}$, $\eta_i = \frac{H_i}{f}$, $\hat{\eta}_i = \frac{\hat{H}_i}{f}$, $i = 1, 2, \cdots, K$, Eqs.(34) become the following nonlinear equations

$$u_t + \beta(t)u_{xxx} + 6\beta(t)uu_x + \gamma(t)\int_{-\infty}^{x} u_{yy}dx - 24\beta(t)(\rho\sigma)_x = -\sum_{i=1}^{K}(\xi_i\eta_i - \hat{\xi}_i\hat{\eta}_i)_x,$$

$$\rho_t - 2\beta(t)(\rho_{xxx} + 3u\rho_x) - 2\sqrt{3\beta(t)\gamma(t)}(\rho_{xy} + \rho\int_{-\infty}^{x} u_y dx) = \frac{1}{2}\sum_{i=1}^{K}(\hat{\xi}_{ix}\eta_i - \hat{\xi}_i\eta_{ix}),$$

$$\sigma_t - 2\beta(t)(\sigma_{xxx} + 3u\sigma_x) + 2\sqrt{3\beta(t)\gamma(t)}(\sigma_{xy} + \sigma\int_{-\infty}^{x} u_y dx) = \frac{1}{2}\sum_{i=1}^{K}(\hat{\eta}_{ix}\xi_i - \hat{\eta}_i\xi_{ix}),$$

$$\xi_{ixx} + \xi_{iy} = 2\rho\hat{\eta}_i - u\xi_i, \qquad \hat{\xi}_{ixx} + \hat{\xi}_{iy} = 2\rho\eta_i - u\hat{\xi}_i, \qquad i = 1, 2, \cdots, K,$$

$$\eta_{ixx} - \eta_{iy} = 2\sigma\hat{\xi}_i - u\eta_i, \qquad \hat{\eta}_{ixx} - \hat{\eta}_{iy} = 2\sigma\xi_i - u\hat{\eta}_i, \qquad i = 1, 2, \cdots, K,$$

which is just the coupled system of the variable-coefficient KP equation with self-consistent sources.

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