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Remarks on Semiprime Rings with Generalized Derivations

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Abstract

Let R be a ring with centre $Z(R)$. An additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$ (the map d is called the derivation associated with F). In the present note we prove that if a semiprime ring R admits a generalized derivation F, d is the nonzero associated derivation of F, satisfying certain polynomial constraints on a nonzero ideal I, then R contains a nonzero central ideal.

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1. Introduction

Throughout the paper R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies that $a = 0$). We shall write for any pair of elements $x, y \in R$ $[x, y]$, the commutator $xy - yx$. We define $x \circ y$ as $xy + yx$ for all $x, y \in R$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Many algebraist studied generalized derivation in the context of algebras on certain normed spaces (see [4] for reference). By a generalized derivation on an algebra A one usually means a map of the form $x \mapsto ax + xb$ where a and b are fixed elements in A. We prefer to call such maps generalized inner derivations for the reason they present a generalization of the concept of inner derivation (i.e. the map $x \mapsto ax - xb$). Now in a ring, let F be a generalized inner derivation given by $F(x) = ax + xb$. Notice that $F(xy) = F(x)y + xI_b(y)$, where $I_b(y) = yb - by$ is an inner derivation. Motivated by these observations, Bresar [2] introduced the notion of a generalized derivation in rings. More specifically an additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left multiplier (i.e. additive map satisfying $f(xy) = f(x)y)$ for all $x, y \in R$. Some results on generalized derivation can be found in [4]. There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Several authors viz. [1], [3], [7] and [8] have obtained commutativity of prime and semiprime rings satisfying certain polynomial constraints on suitable subsets of rings. In [3] Daif and Bell showed that if R is a semiprime ring, J is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) - [x, y] \in Z(R)$ for all $x, y \in J$, then $J \subseteq Z(R)$.

More recently Shuliang [8] proved that: Let R be a prime ring and U be a Lie ideal such that $u^2 \in U$ for all $u \in U$. If F is a generalized derivation with associated derivation d, then either $d = 0$ or $U \subseteq Z(R)$, satisfying one of the following holds: (i) $d(x) \circ F(y) = 0$, (ii) $d(x) \circ F(y) \mp [x, y] = 0$, (iii) $d(x) \circ F(y) \mp x \circ y = 0$, (iv) $d(x) \circ F(y) \mp xy = 0$, (v) $[d(x), F(y)] = 0$, (vi) $[d(x), F(y)] \mp x \circ y = 0$, (vii) $[d(x), F(y)] \mp [x, y] = 0$, (viii) $[d(x), F(y)] \mp xy = 0$, for all $x, y \in U$. Following this line of investigation our aim is to extend the above cited results of Shuliang [8] to left (or two sided) ideals of semiprime rings.

2. Preliminaries

We begin with the following lemmas which are essential for the development of our main results. The following fact is well known (see for example Chapter 4 in [5]):

Fact 1. Let R be a semiprime ring and P be a nonzero prime ideal of R. If $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$.

Lemma 1. [1] Let R be a semiprime ring and I be a nonzero left ideal of R. If d is a nonzero derivation such that d is centralizing on I, then R contains a nonzero central ideal.

Lemma 2. [6] Let R be a semiprime ring and I be a nonzero left ideal of R. Let d be a derivation on R. If for some positive integers $t_0, t_1, t_2, ...t_n$ and all $x \in I$, the identity $[[...[d(x^{t_0}), x^{t_1}].....], x^{t_n}] = 0$ holds, then either $d(I) = 0$ or else $d(I)$ and $d(R)I$ are contained in a nonzero central ideal of R. In particular when R is a prime ring, R is commutative.

Lemma 3. Let R be a 2-torsion free semiprime ring and J be a nonzero ideal of R. If d is a nonzero derivation such that $Jd^2(J) = (0)$, then $J \subseteq Z(R)$.

Proof. Consider $xd^2(y) = 0$ for all $x, y \in J$. Replacing y by yz and using torsion condition to get

$$xd(y)d(z) = 0 \text{ for all } \ x, y, z \in J. \tag{1}$$

Substitute yu for y in (1) yields that $2xd(y)ud(z) = 0$ for all $x, y, u, z \in J$. Since R is 2-torsion free, we have

$$xd(y)ud(z) = 0 \text{ for all } \ x, y, u, z \in J. \tag{2}$$

Replacing u by rx and y and z by x, we get $xd(x)rxd(x) = 0$ for all $x \in J$ and $r \in R$. Since R is semiprime we obtain

$$xd(x) = 0 \text{ for all } \ x \in J. \tag{3}$$

Again replacing u and x by xr, left multiplying by $d(x)$ and right multiplying by x (2), we get $d(x)xrd(x)xrd(x)x = 0$ for all $x \in J$ and for all $r \in R$. Using semiprimeness of R, we have

$$d(x)x = 0 \text{ for all } \ x \in J. \tag{4}$$

Subtracting (3) and (4), we have $[d(x), x] = 0$ for all $x \in J$. Application of Lemma 1 completes the proof.

3. Main Results

Theorem 1. Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a generalized derivation with associated derivation d such that $d(x) \circ F(y) = 0$ for all $x, y \in I$, then R contains a nonzero central ideal unless $Id(I) = (0)$.

Proof. We have $d(x) \circ F(y) = 0$ for all $x, y \in I$. Replacing y by yz we have $d(x) \circ yd(z) = 0$ for all $x, y, z \in I$. This implies that

$$d(x)yd(z) = -yd(z)d(x) \quad \text{for all } x, y, z \in I. \tag{5}$$

Replacing y by ry in (5) and using (5), we get

$$[d(x), r]yd(z) = 0 \quad \text{for all } x, y, z \in I, \quad r \in R. \tag{6}$$

Substitute xr for r in (6) yields that $[d(x), x]Ryd(z) = 0$ for all $x, y, z \in I$. Now take a family $\{P_\alpha\}$ of prime ideals such that $\cap P_\alpha = (0)$. Fact 1 implies that either $[d(x), x] \in P_\alpha$ for all $x \in I$ or $Id(I) \subseteq P_\alpha$. If $Id(I) \subseteq P_\alpha$ for all α, then $Id(I) \subseteq \cap P_\alpha = (0)$ i.e. $Id(I) = (0)$ a contradiction. Therefore, we have $[d(x), x] \in \cap P_\alpha = (0)$ for all $x \in I$ i.e. $[d(x), x] = 0$ for all $x \in I$. Hence conclusion follows from Lemma 1.

Theorem 2. Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a generalized derivation with associated derivation d such that $d(x) \circ F(y) \mp x \circ y = 0$ for all $x, y \in I$, then R contains a nonzero central ideal unless $Id(I) = (0)$.

Proof. Consider

$$d(x) \circ F(y) - x \circ y = 0 \quad \text{for all } x, y \in I. \tag{7}$$

Replacing y by yz in (7) and using (7), we obtain

$$d(x)yd(z) = -yd(z)d(x) \quad \text{for all } x, y, z \in I. \tag{8}$$

Replacing y by ry in (8) and use (8) to get $[d(x), r]yd(z) = 0$ for all $x, y, z \in I$ and $r \in R$. Substitute xr for r in the last relation, we find that $[d(x), x]Ryd(z) = 0$ for all $x, y, z \in I$. Repeating same arguments as we have used in Theorem 1, we get either $[d(x), x] = 0$ or $yd(z) = 0$ for all $x, y, z \in I$. Since $Id(I) \neq (0)$, we have $[d(x), x] = 0$ for all $x \in I$. Hence, conclusion follows from Lemma 1.

Theorem 3. Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a generalized derivation with associated derivation d such that $d(x) \circ F(y) \mp [x, y] = 0$ for all $x, y \in I$, then R contains a nonzero central ideal unless $Id(I) = (0)$.

Proof. We have

$$d(x) \circ F(y) - [x, y] = 0 \quad \text{for all } x, y \in I. \tag{9}$$

Replacing y by yz in (9) and using (9), we arrive at $d(x) \circ yd(z) - y[x, z] = 0$ for all $x, y, z \in I$. In particular, if we replace z by x we have $d(x) \circ yd(x) = 0$ for all $x, y \in I$. i.e.

$$d(x)yd(x) = -yd(x)d(x) \quad \text{for all } x, y \in I. \tag{10}$$

Replace y by ry in (10) and use (10) to get $[d(x), r]yd(x) = 0$ for all $x, y \in I$ and $r \in R$. Substitute xr for r in the last relation we obtain $[d(x), x]Ryd(x) = 0$ for all $x, y \in I$. Repeating same arguments as we have done in Theorem 1, we get either $[d(x), x] = 0$ or $yd(x) = 0$ for all $x, y \in I$. Since $Id(I) \neq (0)$, we have $[d(x), x] = 0$ for all $x \in I$. Using Lemma 1, R contains a nonzero central ideal.

Theorem 4. Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a generalized derivation with associated derivation d such that $d(x) \circ F(y) \mp xy = 0$ for all $x, y \in I$, then R contains a nonzero central ideal unless $Id(I) = (0)$.

Proof. Consider $d(x) \circ F(y) - xy = 0$ for all $x, y \in I$. Replacing y by yz, we have $d(x) \circ yd(z) = 0$ for all $x, y, z \in I$. i.e.

$$d(x)yd(z) = -yd(z)d(x) = 0 \quad \text{for all } x, y, z \in I. \tag{11}$$

Replacing y by ry in (11) and using (11), we find

$$[d(x), r]yd(z) = 0 \quad \text{for all } x, y, z \in I, \quad r \in R. \tag{12}$$

Substitute xr for r in (12) yields that $[d(x), x]Ryd(z) = 0$ for all $x, y, z \in I$. Using same techniques as we have done in the proof of Theorem 1, we get either $[d(x), x] = 0$ or $yd(z) = 0$ for all $x, y, z \in I$. Since $Id(I) \neq (0)$, we have $[d(x), x] = 0$ for all $x \in I$. Hence conclusion follows from Lemma 1.

Theorem 5. Let R be a 2-torsion free semiprime ring and J be a nonzero ideal of R. If F is a generalized derivation with associated derivation d such that $[d(x), F(y)] = 0$ for all $x, y \in J$, then $J \subseteq Z(R)$.
Proof. Let

$$[d(x), F(y)] = 0 \quad \text{for all } x, y \in J. \tag{13}$$

Replacing x by xz in (13), we get $[d(x), F(y)]z + d(x)[z, F(y)] + x[d(z), F(y)] + [x, F(y)]d(z) = 0$ for all $x, y, z \in J$. Application of (13) yields that

$$d(x)[z, F(y)] + [x, F(y)]d(z) = 0 \quad \text{for all } x, y, z \in J. \tag{14}$$

Again replacing z by $zd(x)$ in (14) we have $d(x)[z, F(y)]d(x) + [x, F(y)]d(z)d(x) + [x, F(y)]zd^2(x) = 0$ for all $x, y, z \in J$. Using (14) we obtain

$$[x, F(y)]zd^2(x) = 0 \quad \text{for all } x, y, z \in J. \tag{15}$$

Substitute rz for z in (15) to get $[x, F(y)]Rzd^2(x) = 0$ for all $x, y, z \in J$. Repeating the same arguments as we have done in Theorem 1, we get either $[F(y), x] = 0$ or $zd^2(x) = 0$ for all $x, y, z \in J$. If $Jd^2(J) = 0$, then conclusion follows from Lemma 3. If $[F(y), x] = 0$ for all $x, y \in J$, then we get by replacing y by yx

$$y[d(x), x] + [y, x]d(x) = 0 \quad \text{for all } x, y \in J. \tag{16}$$

Replacing y by ry in (16) and using (16), we have

$$[r, x]yd(x) = 0 \quad \text{for all } x, y \in J, \quad r \in R. \tag{17}$$

Substitute yx for y in (17) yields that

$$[r, x]yxd(x) = 0 \quad \text{for all } x, y \in J, \quad r \in R. \tag{18}$$

Left multiplying by x in (17) to get

$$[r, x]yd(x)x = 0 \quad \text{for all } x, y \in J, \quad r \in R. \tag{19}$$

Subtracting (18) and (19), we have

$$[r,x]y[d(x),x]=0 \quad \text{for all } x,y\in J, \quad r\in R. \tag{20}$$

Substitute $d(x)$ for r and rx for y and left multiplying by x to (20) yields that $x[d(x),x]rx[d(x),x]=0$ for all $x\in J$ and for all $r\in R$. Since R is semiprime, we have

$$x[d(x),x]=0 \quad \text{for all } x\in J. \tag{21}$$

Substitute $d(x)$ for r and xr for y and right multiplying by x to (20) yields that $[d(x),x]xr[d(x),x]x=0$ for all $x\in J$ and for all $r\in R$. Using semiprimeness of R we find that

$$[d(x),x]x=0 \quad \text{for all } x\in J. \tag{22}$$

Subtracting (21) and (22) to get $[[d(x),x],x]=0$ for all $x\in J$. Application of Lemma 2 gives the required result.

Theorem 6. Let R be a semiprime ring and J be a nonzero ideal of R. If F is a generalized derivation with associated derivation d such that $[d(x),F(y)]\mp x\circ y=0$ for all $x,y\in J$, then R contains a nonzero central ideal.

Proof. Consider

$$[d(x),F(y)]-x\circ y=0 \quad \text{for all } x,y\in J. \tag{23}$$

Replacing y by yz in (23), we have $[d(x),F(y)]z+F(y)[d(x),z]+y[d(x),d(z)]+[d(x),y]d(z)-x\circ yz=0$ for all $x,y,z\in J$. This implies that

$$F(y)[d(x),z]+y[d(x),d(z)]+[d(x),y]d(z)=0 \quad \text{for all } x,y,z\in J. \tag{24}$$

Replacing z by $d(x)z$ in (24) to get

$$F(y)d(x)[d(x),z]+yd^2(x)[d(x),z]+y[d(x),d^2(x)]z+yd(x)[d(x),d(z)]$$
$$+[d(x),y]d^2(x)z+[d(x),y],d(x)]d(z)=0 \quad \text{for all } x,y,z\in J. \tag{25}$$

Multiplying (24) by $d(x)$ from left and subtracting from (25), we obtain

$$[d(x),F(y)][d(x),z]+yd^2(x)[d(x),z]+[y,d(x)][d(x),d(z)]$$
$$+[d(x),y]d^2(x)z+[[d(x),y],d(x)]d(z)=0 \quad \text{for all } x,y,z\in J. \tag{26}$$

Application of (23) yields that

$$xy[d(x),z]+yx[d(x),z]+yd^2(x)[d(x),z]+[y,d(x)][d(x),d(z)]$$
$$+[d(x),y]d^2(x)z+[[d(x),y],d(x)]d(z)=0 \quad \text{for all } x,y,z\in J. \tag{27}$$

Substitute ry for y in (27) to get

$$xry[d(x),z]+ryx[d(x),z]+ryd^2(x)[d(x),z]+r[y,d(x)][d(x),d(z)]$$
$$+[d(x),r]yd^2(x)z+[r,d(x)]y[d(x),d(z)]+r[d(x),y]d^2(x)z+2[d(x),r][y,d(x)]d(z)$$
$$+[[d(x),r],d(x)]yd(z)+r[[d(x),y],d(x)]d(z)=0 \text{ for all } x,y,z\in I,\ r\in R. \tag{28}$$

In view of (27), (28) reduces to

$$[x,r]y[d(x),z]+[r,d(x)]y[d(x),d(z)]+2[d(x),r][y,d(x)]d(z)$$
$$+[[d(x),r],d(x)]yd(z)+[d(x),r]yd^2(x)z=0 \quad \text{for all } x,y,z\in J,\ r\in R. \tag{29}$$

Substitute $d(x)$ for r in (29) to get

$$[x,d(x)]y[d(x),z]=0 \quad \text{for all } x,y,z\in J. \tag{30}$$

Again replacing y by rx and z by x and left multiplying by x in (30), we get by using semiprimeness of R

$$x[x,d(x)]=0 \quad \text{for all } x\in J. \tag{31}$$

If we replace y by xr, z by x and right multiplying by x in (30), then semiprimeness of R yields that

$$[x,d(x)]x=0 \quad \text{for all } x\in J. \tag{32}$$

Subtracting (31) and (32), we have $[x,[x,d(x)]]=0$ for all $x\in J$. Hence conclusion follows from Lemma 2.

Theorem 7. Let R be a 2-torsion free semiprime ring and J be a nonzero ideal of R. If F is a generalized derivation with associated derivation d such that $[d(x),F(y)]\mp[x,y]=0$ for all $x,y\in J$, then R contains a nonzero central ideal.

Proof. Suppose that

$$[d(x),F(y)]-[x,y]=0 \quad \text{for all } x,y\in J. \tag{33}$$

Replacing x by xz in (33), we have $d(x)[z,F(y)]+[d(x),F(y)]z+x[d(z),F(y)]+[x,F(y)]d(z)-x[z,y]-[x,y]z=0$ for all $x,y,z\in J$. This implies that $d(x)[z,F(y)]+[x,F(y)]d(z)=0$ for all $x,y,z\in J$. Repeating the same techniques as we have done in Theorem 5, we get the required result.

Theorem 8. Let R be a semiprime ring and J be a nonzero ideal of R. If F is a generalized derivation with associated derivation d such that $[d(x),F(y)]\mp xy=$

0 for all $x, y \in J$, then R contains a nonzero central ideal.

Proof. Consider $[d(x), F(y)] - xy = 0$ for all $x, y \in J$. Replacing y by yz in the last relation we have $[d(x), F(y)]z + F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) - xyz = 0$ for all $x, y, z \in J$. This implies that $F(y)[d(x), z] + y[d(x), d(z)] + [d(x), y]d(z) = 0$ for all $x, y, z \in J$. Following the outline of the proof of Theorem 6, we arrive at the conclusion.

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