International Mathematical Forum, Vol. 7, 2012, no. 26, 1263 - 1278

Dual Motions and Real Spatial Motions

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Abstract

In this study, by taking two dual unit spheres moving with respect to each other, we define different motion on the system. It shows us that there is a big relation between dual motions and spatial motions. By the help of curve-surface frame, we have obtained another frame on the unit dual sphere and then the motion of this frame is investigated. In addition, by presenting the ruled surfaces that correspond to these dual curves, we give the special cases for them and show that when they could be developable or not. Moreover, some examples and results are presented.

Mathematics Subject Classification: 53A04, 53A05

Keywords: Curvature, Geodesic torsion, Ruled surface, Spherical dual curve

1 Introduction

Dual numbers had been introduced by Clifford as a tool for his geometrical investigations. After him, E. Study used dual numbers and dual vectors in his research on line geometry and kinematics [6,11].

In spatial kinematics, the geometry of infinitesimally seperated positions of a system of rigid bodies is a diffucult but important and interesting subject that has been widely studied during the past decades. It is also well known in spatial kinematics that the instantaneous screw axis (ISA) at a prescribed instant can be determined by the first order derivatives of a conjugate motion of one degree of freedom. [1]

In a spatial motion, the trajectory of the oriented lines and points embedded in a moving rigid body are generally ruled surfaces and curves, respectively [7]. G. R. Veldkamp investigated some spatial and spherical problems by using the geometry of curves and developable ruled surfaces in [11]. In the same paper, he also gave the relation between dual motion and spatial motion.

In this study, using the same way as in [11], we show that the motion of the dual curve gives us the surface when the motion of the point gives us the line, on dual sphere. Besides, we investigate when the ruled surfaces could be developable or not. According to this, the special cases for the curves which are geodesic or curvature line are presented.

2 BASIC CONCEPTS

Now, we recall some basic concepts of the differential geometry of curves and surfaces:

2.1 The Darboux Frame, The Geodesic Curvature and Geodesic Torsion [8]

Let $\alpha : I \to M$ be unit speed curve whose support lies on a surface M and let $\alpha(s) = p$ be a point of the curve. The new orthonormal frame which involves both vectors related to the curve and to the surface should be introduced. The first vector of the new frame will be still the unit tangent vector of the curve, T. The second, related, this time to the surface, is the unit normal of the surface, Y. The third one, let it be denoted by N, will be chosen in such a way that the basis $\{T, Y, N\}$ be direct, or in other words, such that $(T, Y, N) = 1$. This means, of course, that

$$Y = N \times T \tag{1}$$

Clearly, N lies in the normal plane of the curve, therefore it will be called the unit tangential normal vector of he curve at p. The name tangential comes, of course from the fact that N lies, also, in the tangent plane of the surface at p.

The frame $\{T, Y, N\}$ is called the Darboux frame or the Ribaucour-Darboux frame of the surface M along the curve α.

The next step we are going to make is to compute the derivatives of the vectors of the Darboux frame and to obtain a set of linear differential equation which is smiliar to the Frenet Frame. To get them, the intention is exactly to use the Frenet equations. Therefore we shall start by expressing the vectors Y

and N in terms of the vectors of the Frenet frame. We remind that the derivative of a vector the Darboux frame is pependicular on that vector, because the frame is orthonormal. For the same reason, we deduce immediately that the coefficients are not dependent and, in fact, we call them the curvatues of the curve-surface pair (α, M) [4, 5, 9]. Thus the system becomes as following:

$$\begin{aligned} T' &= k_g Y + k_n N \\ Y' &= -k_g T + t_r N \\ N' &= -k_n T - t_r Y \end{aligned} \tag{2}$$

2.2 Curvature Line [3]

A curve all of whose tangents are in principal directions is a curvature line. The normal curvature of a curvature line is a principal curvature of a surface. A point at which two principal curvatures coincide is an umbilic.

Theorem 2.1 *The oriented lines in R^3 are in one to one correspondance with the points of the dual unit sphere $\langle A, A\rangle = 1$ in D^3.*[6, 10]

3 THE RELATION BETWEEN DUAL MOTION AND REAL SPATIAL MOTION [11]

We consider oncemore the dual motion $\widehat{X} = \widehat{A}\widehat{x}$, where $\widehat{A} = A + \varepsilon A^* = (\alpha_{ik}) + \varepsilon(\alpha^*_{ik})$ is a direct orthogonal dual 3×3 matrix. There exists an unambiguously determined vector α such that $\alpha^*_{.k} = \alpha \times \alpha_{.k} (k = 1, 2, 3)$; obviously α is a function of t. Putting $\alpha^T = (\alpha_1, \alpha_2, \alpha_3)$ we have

$$A^* = (\alpha^*_{ik}) = \begin{bmatrix} a_2\alpha_{31} - \alpha_3\alpha_{21} & a_2\alpha_{32} - \alpha_3\alpha_{22} & a_2\alpha_{33} - \alpha_3\alpha_{23} \\ a_3\alpha_{11} - \alpha_1\alpha_{31} & a_3\alpha_{12} - \alpha_2\alpha_{32} & a_3\alpha_{13} - \alpha_1\alpha_{33} \\ a_1\alpha_{21} - \alpha_2\alpha_{11} & a_1\alpha_{22} - \alpha_2\alpha_{12} & a_1\alpha_{23} - \alpha_2\alpha_{13} \end{bmatrix} \tag{3}$$

Let $\widetilde{\delta} == \delta + \varepsilon\delta^*$ be a point on the moving sphere$\widehat{K}$. There exists a unique π such that $\delta^* = \pi \times \delta, (\pi, \delta) = 0$. The point $\widetilde{\delta}$ corresponds under the mapping

$$[x, x^*] \Leftrightarrow x + \varepsilon x^* = \widehat{x}. \tag{4}$$

to the spear $[\delta, \delta^*]$ with $x = \pi + \lambda\delta$ as its carrier. We have:

$$\begin{aligned} \widehat{A}\widetilde{\delta} &= (A + \varepsilon A^*)(\delta + \varepsilon\delta^*) \\ &= A\delta + \varepsilon(A^*\delta + A\delta^*) \\ &= A\delta + \varepsilon(A^*\delta + A\pi \times A\delta). \end{aligned} \tag{5}$$

Putting $\delta^T = (\delta_1, \delta_2, \delta_3)$ we obtain:

$$A^*\delta = \begin{bmatrix} \alpha_2(\alpha_{3.}, \delta) - \alpha_3(\alpha_{2.}, \delta) \\ \alpha_3(\alpha_{1.}, \delta) - \alpha_1(\alpha_{3.}, \delta) \\ \alpha_1(\alpha_{2.}, \delta) - \alpha_2(\alpha_{1.}, \delta) \end{bmatrix} = \alpha \times A\delta. \tag{6}$$

Hence: $\widehat{A}\widetilde{\delta} = A\delta + \varepsilon(A\pi + \alpha) \times A\delta$. This shows that $\widehat{A}\widetilde{\delta}$ corresponds to the spear $[A\delta, (A\pi + \alpha) \times A\delta]$ having

$$X = A\pi + \alpha + \lambda A\delta \tag{7}$$

as its carrier. Since A and α are functions of t, (7) represents a ruled surface for variable t and λ. This surface is obviously generated by the line

$$x = \pi + \lambda A\delta, (\delta, \delta) = 1, (\pi, \delta) = 0 \tag{8}$$

under the motion

$$X = Ax + \alpha \tag{9}$$

of a Euchlidean space E_2 containing this line with respect to a Euchlidean space E_1.

Besides we can show that

$$\begin{aligned} (A + \varepsilon A^*)(A^T + \varepsilon A^{*T}) &= I \\ &= AA^T + \varepsilon(AA^{*T} + A^*A^T) \end{aligned} \tag{10}$$

Here, when we get

$$AA^{*T} = -A^*A^T \tag{11}$$

where

$$D = A^*A^T \text{ and} D^T \ = AA^{*T} \tag{12}$$

we can easily obtain that

$$D^T = -D \tag{13}$$

This means that D is antisymetric.

At this time, let get direct orthogonal dual 3×3 matrix $\widehat{A}$ as

$$\widehat{A} = [\widehat{T} \ \widehat{Y} \ \widehat{N}] \tag{14}$$

to investigate the instantaneous screw motion on dual sphere in the next section.

3.0.1 DUAL UNIT MATRICES AND DUAL MOTIONS

Let

$$\begin{aligned} \alpha : I &\to E^3 \\ s &\mapsto \alpha(s) \end{aligned} \tag{15}$$

be unit speed curve on the surface M in E^3 and T be the unit tangent vector field of α. Because of the properties of vector product, the orthonormal frame $\{T, Y, N\}$ is an orthonormal base of T_pR^3) such that $s \in I$ and also $\alpha(s) = p$. In that case, this orthonormal frame is called curve-surface frame. With the assistance of this frame, it can be easily said that (α, M) is curve-surface pairs.

In this section, with the assistance of α, we define a dual curve on another surface $\widetilde{M}$ in D^3. So, let us have a closed spherical dual curve $\widehat{\alpha}$ of class C^1 on the surface $\widetilde{M}$, on the unit dual sphere S^1in D^3. The curve $\widehat{\alpha}$ describes a closed dual spherical motion. If

$$\begin{aligned} \widehat{\alpha} : \quad & I \to D^3 \\ & s \mapsto \widehat{\alpha}(s) = \alpha(s) + \varepsilon \int \alpha \wedge T ds = \int (T + \varepsilon(\alpha \wedge T)) ds \end{aligned} \tag{16}$$

then the curve $\widehat{\alpha}$ can be written as :

$$\widehat{\alpha} = \alpha(s) + \varepsilon \alpha^{\star}(s). \tag{17}$$

Besides, we can easily show that the curves α and $\widehat{\alpha}$ have the same parameter s from [12]

On the other hand, let K and $\widehat{K}$ be dual unit spheres moving with respect to each other. These are two orthonormal coordinate systems of moving unit sphere K and fixed unit dual sphere $\widehat{K}$ with the same origin. We may represent the dual motion with $\widehat{K}/K$. We suppose that the orthonormal trihedron which are being used to represent the motions coincide at the instant under consideration.

On this spherical motion that is given with the help of dual matrice $\widehat{A}(t)$, the direct(yörünge) of the fixed point $\overline{\delta} \in \widehat{K}$ is given with spherical curve as follows:

$$\overline{X}(t) = \widehat{A}(t)\overline{\delta} \tag{18}$$

The ruled surface that corresponds to this spherical curve $\overline{X}(t)$ can be found by these calculations as follows:

$$\begin{aligned} \widehat{A\overline{\delta}} &= \widehat{A}(\delta + \varepsilon \pi \times \delta) \\ &= A\delta + \varepsilon((A\pi + \alpha) \times A\delta) \\ &= \alpha(t) + \varepsilon \alpha^*(t) \end{aligned} \tag{19}$$

Thus we obtain

$$\Phi(\lambda, s) = (A\pi + \alpha) + \lambda A\delta \tag{20}$$

as its carrier. Since A and α are the functions of s, (20) represents a ruled surface for variable s and λ. The surface is obviously generated by the line

$$x = \pi + \lambda\delta,\ (\delta, \delta) = 1,\ (\pi, \delta) = 0 \tag{21}$$

under the motion

$$X = Ax + \alpha \tag{22}$$

of an Euchlidean space E_2 containing this line with respect to an Euchlidean space E_1. This gives us the joint and relation between real spatial motion and dual motion.

Here, it can be easily shown that the ruled surface is developable if

$$\begin{aligned} \det(A'\pi + T, A\delta, A'\delta) &= 0 \\ \det(A'\pi, A\delta, A'\delta) + \det(T, A\delta, A'\delta) &= 0 \\ \langle A'\pi, A\delta \times A'\delta\rangle + \langle T, A\delta \times A'\delta\rangle &= 0 \\ \langle A'\pi + T, A\delta \times A'\delta\rangle &= 0 \end{aligned} \tag{23}$$

Accordingly, the ruled surface of the line that passes through the origin $x = \lambda\delta$ is

$$\Phi(\lambda, s) = \alpha(s) + \lambda A\delta \tag{24}$$

Thus the ruled surface is developable if and only if

$$\det(T, A\delta, A'\delta) = 0 \tag{25}$$

On the other hand, let A be the direct orthogonal dual 3×3 matrice. Thus, it can be easily shown that

$$A' = [T'\ Y'\ N'] \tag{26}$$

We can easily say that the surface that is drawn by the point $\delta = (1, 0, 0)$ gives us the the curve $\widehat{T}$. Smiliarly It can be said for the other points too. These situations can be given as examples as follows:

$$\begin{aligned} \widehat{A} &= \left[\widehat{T}\ \widehat{Y}\ \widehat{N}\right] = A + \varepsilon A^* \\ &= [T + \varepsilon\alpha \wedge T\ Y + \varepsilon\alpha \wedge Y\ N + \varepsilon\alpha \wedge N] \end{aligned} \tag{27}$$

$$\Phi(\lambda, s) = A\pi + \alpha(s) + \lambda A\delta \tag{28}$$

1. If $\pi = (0, 0, 0)$ and $\delta = (1, 0, 0)$, then the ruled surface $\Phi_{\widehat{T}} = \alpha(s) + \lambda T$ is obtaned.

2. If $\pi = (0,0,0)$ and $\delta = (0,1,0)$, then the ruled surface $\Phi_{\widehat{Y}} = \alpha(s) + \lambda Y$ is obtaned.

3. If $\pi = (0,0,0)$ and $\delta = (0,0,1)$, then the ruled surface $\Phi_{\widehat{N}} = \alpha(s) + \lambda N$ is obtaned.

Thus, the ruled surfaces that correspond to dual curves $\widehat{T}, \widehat{Y}$ and $\widehat{N}$ are the ruled surfaces that drawn by the lines T, Y and N of α. That is to say, these ruled surfaces can be given as:

$$\begin{aligned} \Phi_{\widehat{T}} &= \alpha(s) + \lambda T(s) \\ \Phi_{\widehat{Y}} &= \alpha(s) + \lambda Y(s) \\ \Phi_{\widehat{N}} &= \alpha(s) + \lambda N(s) \end{aligned} \tag{29}$$

Accordingly, we define an orthonormal moving frame $\left\{\widehat{T}, \widehat{Y}, \widehat{N}\right\}$ along dual curve in D^3 as follows:

$$\widehat{\alpha}(s) = \int [T(s) + \varepsilon\alpha(s) \wedge T(s)]ds, \tag{30}$$

$$\widehat{\mu}(s) = \int [Y(s) + \varepsilon\alpha(s) \wedge Y(s)]ds \tag{31}$$

$$\widehat{\gamma}(s) = \int (N(s) + \varepsilon\alpha(s) \wedge N(s))ds \tag{32}$$

And also, let the vector fields of $\widehat{\alpha}$ be $\widehat{T}(s)$, $\widehat{Y}(s)$ and $\widehat{N}(s)$ as:.

$$\begin{aligned} \widehat{T}(s) &= T + \varepsilon(\alpha \wedge T), \\ \widehat{Y}(s) &= Y + \varepsilon(\alpha \wedge Y), \\ \widehat{N}(s) &= N + \varepsilon(\alpha \wedge N), \end{aligned} \tag{33}$$

Subsequently, we can give the following theorem:

Theorem 3.1 *The curves $\widehat{\alpha}(s)$, $\widehat{\mu}(s)$ and $\widehat{\gamma}(s)$ are involute-evolute curve pairs [12].*

Proof. It can be easily seen that

$$\langle \widehat{\alpha}'(s), \widehat{\mu}'(s) \rangle = \langle \widehat{\alpha}'(s), \widehat{\gamma}'(s) \rangle = \langle \widehat{\mu}'(s), \widehat{\gamma}'(s) \rangle = 0 \tag{34}$$

Now, we can investigate the developability of the ruled surfaces that correspond to vector fields $\widehat{T}, \widehat{Y}$ and $\widehat{N}$ of the dual curve $\widehat{\alpha}$. If $\Phi_{\widehat{T}} = \alpha(s) + \upsilon T(s)$, then

$$\begin{aligned} P_{\widehat{T}} &= \frac{\det(\alpha', T, T')}{\|T'\|^2} \\ &= \frac{\det(T, T, k_g Y + k_n N)}{k_g^2 + k_n^2} \\ &= 0 \end{aligned} \tag{35}$$

Therefore, the ruled surface is developable.

After that, it can be easily seen that if $\Phi_{\widehat{Y}} = \alpha(s) + \upsilon Y(s)$, then

$$\begin{aligned} P_{\widehat{Y}} &= \frac{\det(\alpha', Y, Y')}{\|Y'\|^2} \\ &= \det(T, Y, -k_g T + t_r N) \\ &= t_r \end{aligned} \tag{36}$$

Thus, if $P_{\widehat{Y}}$ is zero then the ruled surface is developable. At that time t_r is also zero, so if $t_r = 0$, then the curve $\widehat{\alpha}$ is curvature line. That is to say if the geodesic torsion is zero, the ruled surface is developable.

Result. The ruled surface $\Phi_{\widehat{Y}}$ is developable if and only if the curve α is a curvature line on the surface M.

Example 3.2 *Let $\alpha : I \to S^2$ be a curve and suppose that we have the frame $\{T, Y, N\}$. Here, $P_{\widehat{Y}} = t_r = 0$. Thus a curve on a sphere is a curvature line.*

Subsequently, if $\Phi_{\widehat{N}} = \alpha(s) + \upsilon N(s)$, then

$$\begin{aligned} P_{\widehat{N}} &= \det(\alpha', N, N') \\ &= \det(T, N, -k_n T - t_r Y) \\ &= t_r \end{aligned} \tag{37}$$

Here, if the geoesic torsion of $(\widehat{\alpha}, \widetilde{M})$ is zero, then the ruled surface is developable, If $t_r = 0$ then the curve $\widehat{\alpha}$ is a curvature line.

Result. Smiliarly, the ruled surface $\Phi_{\widehat{N}}$ is developable if and only if the curve α is a curvature line on the surface M.

Result. It can be easily seen that $k_n = k_g$ if and only if $P_{\widehat{Y}} = P_{\widehat{N}}$.

On the other hand, let the unit dual spheres K and $\overline{K}$ be

$$K = \{e_1, e_2, e_3\} \text{ and } \overline{K} = \left\{\widehat{T}, \widehat{Y}, \widehat{N}\right\} \tag{38}$$

In this case, we can easily investigate one parameter dual spherical motion (dual rotation) $K/\overline{K}$ with

$$\begin{bmatrix} \widehat{T}' \\ \widehat{Y}' \\ \widehat{N}' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_r + \varepsilon \\ -k_n & -t_r - \varepsilon & 0 \end{bmatrix} \begin{bmatrix} \widehat{T} \\ \widehat{Y} \\ \widehat{N} \end{bmatrix} \tag{39}$$

Besides, let the Darboux indicatrice curve of $\widehat{\alpha}(s)$ be $\widehat{W}$. According to this, the Darboux vector of this motion is:

$$\widehat{W} = (t_r + \varepsilon)\widehat{T} - k_n \widehat{Y} + k_g \widehat{N} \tag{40}$$

and

$$\begin{aligned}\left\|\widehat{W}\right\| &= \sqrt{(t_r+\varepsilon)^2+k_n^2+k_g^2} = \sqrt{(t_r^2+k_g^2)+2\varepsilon t_r} \\ &= \sqrt{t_r^2+k_n^2+k_g^2}+\varepsilon\frac{t_r}{\sqrt{t_r^2+k_n^2+k_g^2}} \\ &= \Psi+\varepsilon\Psi^*\end{aligned} \tag{41}$$

Here, taking the curvature of the curve α as

$$k_n^2+k_g^2=\kappa^2$$

the pitch of the instantenous screw motion can be given by:

$$\begin{aligned}\frac{\Psi^*}{\Psi} &= \frac{d\frac{t_r}{\sqrt{t_r^2+\kappa^2}}}{\sqrt{t_r^2+\kappa^2}} \\ &= \frac{t_r}{t_r^2+\kappa^2}\end{aligned} \tag{42}$$

In summary, this is the dual rotation Pfaffian vector. The real part Ψ and dual part Ψ^* correspond to the rotation and translation motions. In order to leave out the pure translation we will suppose that $\Psi \neq 0$.

According to Pfaffian vector, it can be easily said that if $\Psi \neq 0, \Psi^* \neq 0$, then there are rotation and translation motions (helix motion).

And also if $\Psi \neq 0, \Psi^* = 0$ then it can be obtained pure rotation (spherical motion).

Result. When the curve α isn't a curvature line, we have the instantenous screw motion where the pitch is

$$\frac{\Psi^*}{\Psi}=\frac{t_r}{t_r^2+k_n^2+k_g^2} \tag{43}$$

Result. When the curve α is a curvature line that is to say $t_r = 0$, we have $\Psi \neq 0, \Psi^* = 0$ and it can be obtained pure rotation (spherical motion).

The instantenous screw motion is only rotation motion.

Result. If the curve α is geodesic, then the pitch is

$$\frac{\Psi^*}{\Psi}=\frac{t_r}{t_r^2+k_n^2}.$$

Result. If the curve α is asymptotic then the pitch is

$$\frac{\Psi^*}{\Psi}=\frac{t_r}{t_r^2+k_g^2}. \tag{44}$$

3.0.2 SECOND KIND OF DUAL MOTION

In similiar way, with the assistance of α, we can define curve β. Let $\beta(s)$ be unit speed curve and its parameter be the same as the parameter of the curve $\alpha(s)$.

$$\begin{aligned} \beta : \quad & I \to E^3 \\ & s \mapsto \beta(s) \end{aligned} \tag{45}$$

Here, another dual curve $\widehat{\beta}$ on the same surface $\widetilde{M}$ in D^3 can be defined. At this time, we have to say that the frame $\{T, Y, N\}$ is an orthonormal moving frame along dual curves in D^3. Thus, let us have another closed spherical dual curve $\widehat{\beta}$ of class C^1 on the surface $\widetilde{M}$ in D^3. The curve β describes a closed dual spherical motion if

$$\begin{aligned} \widehat{\beta} : \quad & I \to D^3 \\ & s \mapsto \widehat{\beta}(s) = \alpha(s) + \varepsilon \int \beta \wedge T ds = \int (T + \varepsilon(\beta \wedge T)) ds \end{aligned} \tag{46}$$

On the other hand, let K and $\widehat{K}$ be the same dual unit spheres moving with respect to each other. These are two orthonormal coordinate systems of moving unit sphere K and fixed unit dual sphere $\widehat{K}$ with the same origin. We may represent the dual motion with $\widehat{K}/K$. We suppose that the orthonormal trihedron which are being used to represent the motions coincide at the instant under consideration.

On this spherical motion that is given with the help of dual matrice $\widehat{B}(t)$, the orbit of the fixed point $\overline{\delta} \in \widehat{K}$ is given with spherical curve as follows:

$$\overline{P}(t) = \widehat{B}(t)\overline{\delta} \tag{47}$$

The ruled surface that corresponds to this spherical curve $\overline{Y}(t)$ can be found by these calculations as follows:

$$\begin{aligned} \widehat{B}\overline{\delta} &= \widehat{B}(\delta + \varepsilon\pi \times \delta) \\ &= B\delta + \varepsilon((B\pi + \alpha) \times B\delta) \\ &= \beta(t) + \varepsilon\beta^*(t) \end{aligned} \tag{48}$$

Thus we obtain

$$\widetilde{\Phi}(\lambda, s) = (B\pi + \beta) + \lambda B\delta \tag{49}$$

as its carrier. Since B and β are the functions of s, (49) represents a ruled surface for variable s and λ. The surface is obviously generated by the line

$$x = \pi + \lambda\delta, \ (\delta, \delta) = 1, \ (\pi, \delta) = 0 \tag{50}$$

under the motion

$$\widetilde{X} = Bx + \beta \tag{51}$$

of an Euchlidean space E_2 containing this line with respect to an Euchlidean space E_1. This gives us the joint and relation between real spatial motion and dual motion.

On the other hand it can be easily shown that the ruled surface is developable if

$$\begin{aligned} \det(B'\pi + T, B\delta, B'\delta) &= 0 \\ \det(B'\pi, B\delta, B'\delta) + \det(T, B\delta, B'\delta) &= 0 \\ \langle B'\pi, B\delta \times B'\delta\rangle + \langle T, B\delta \times B'\delta\rangle &= 0 \\ \langle B'\pi + T, B\delta \times B'\delta\rangle &= 0 \end{aligned} \tag{52}$$

Accordingly, the ruled surface of the line that passes through the origin $x = \lambda\delta$ is

$$\widetilde{\Phi}(\lambda, s) = \beta(s) + \lambda B\delta \tag{53}$$

Thus the ruled surface is developable if and only if

$$\det(T, B\delta, AB'\delta) = 0 \tag{54}$$

Besides we can show that

$$\begin{aligned} (B + \varepsilon B^*)(B^T + \varepsilon B^{*T}) &= I \\ &= BB^T + \varepsilon(BB^{*T} + B^*B^T) \end{aligned} \tag{55}$$

Here, when we get

$$BB^{*T} = -B^*B^T \tag{56}$$

where

$$\overline{R} = B^*B^T \text{ and } R^T = BB^{*T} \tag{57}$$

we can easily obtain that

$$\overline{R}^T = -\overline{R} \tag{58}$$

This means that R is antisymetric.

On the other hand, let B be the direct orthogonal dual 3×3 matrice. Thus, it can be easily shown that

$$B' = [T', Y', N'] \tag{59}$$

We can easily say that the surface that is drawn by the point $\delta = (1, 0, 0)$ gives us the the curve $\overline{T}$. Smiliarly, It can be said for the other points too. These situations can be given as examples as follows:

$$\begin{aligned} \widehat{B} &= \left[\overline{T} \;\; \overline{Y} \;\; \overline{N}\right] \qquad (60) \\ &= \left[T + \varepsilon\beta \wedge T \;\;\; Y + \varepsilon\alpha \wedge Y \;\;\; N + \varepsilon\beta \wedge N\right] \end{aligned}$$

$$\widetilde{\Phi}(\lambda, s) = B\pi + \beta(s) + \lambda B\delta$$

1. If $\pi = (0, 0, 0)$ and $\delta = (1, 0, 0)$, then the ruled surface $\widetilde{\Phi}_{\overline{T}} = \beta(s) + \lambda T$ is obtaned.
2. If $\pi = (0, 0, 0)$ and $\delta = (0, 1, 0)$, then the ruled surface $\widetilde{\Phi}_{\overline{Y}} = \beta(s) + \lambda Y$ is obtaned.
3. If $\pi = (0, 0, 0)$ and $\delta = (0, 0, 1)$, then the ruled surface $\widetilde{\Phi}_{\overline{N}} = \beta(s) + \lambda N$ is obtaned.

Thus, the ruled surfaces that correspond to dual curves $\overline{T}$, $\overline{Y}$ and $\overline{N}$ are the ruled surfaces that drawn by the lines T, Y and N of β. That is to say, these ruled surfaces can be given as:

$$\begin{aligned} \widetilde{\Phi}_{\overline{T}} &= \beta(s) + \lambda T(s) \qquad (61) \\ \widetilde{\Phi}_{\overline{Y}} &= \beta(s) + \lambda Y(s) \\ \widetilde{\Phi}_{\overline{N}} &= \beta(s) + \lambda N(s) \end{aligned}$$

In that case, taking $\overline{T}, \overline{Y}$ and $\overline{N}$ as the vector fields of the other dual curve $\widehat{\beta} = \int (T + \varepsilon(\beta \wedge T))ds$, we get

$$\begin{aligned} \overline{T} &= T + \varepsilon\beta \wedge T \qquad (62) \\ \overline{Y} &= Y + \varepsilon\beta \wedge Y \\ \overline{N} &= N + \varepsilon\beta \wedge N. \end{aligned}$$

and also ruled surfaces that correspond to these vector areas of the other dual curve $\widehat{\beta}$ can be given as:

$$\begin{aligned} \widetilde{\Phi}_{\overline{T}} &= \beta + vT \qquad (63) \\ \widetilde{\Phi}_{\overline{Y}} &= \beta + vY \\ \widetilde{\Phi}_{\overline{N}} &= \beta + vN \end{aligned}$$

Now, the developability of the ruled surfaces that correspond to $\overline{T}$, $\overline{Y}$ and $\overline{N}$ can be investigated. Firstly, if $\widetilde{\Phi}_{\overline{T}} = \beta + vT$, then

$$\begin{aligned} P_{\overline{T}} &= \frac{\det(\beta', T, T')}{k_g^2 + k_n^2} \qquad (64) \\ &= \frac{\det(\lambda_1 T + \lambda_2 Y + \lambda_3 N, T, k_g Y + k_n N)}{k_g^2 + k_n^2} \\ &= \frac{-\lambda_2 k_n + \lambda_3 k_g}{k_g^2 + k_n^2} \end{aligned}$$

The ruled surface is developable if and only if $P_{\overline{T}} = 0$, and also

$$\frac{\lambda_2}{\lambda_3} = \frac{k_g}{k_n}$$

If $k_g = 0$ then the curve β is geodesic. Secondly, if $\widetilde{\Phi}_{\overline{Y}} = \beta + vY$ then we give

$$\begin{aligned} P_{\overline{Y}} &= \frac{\det(\beta', Y, Y')}{k_g^2 + t_r^2} \\ &= \frac{\det(\lambda_1 T + \lambda_2 Y + \lambda_3 N, Y, -k_g T + t_r N)}{k_g^2 + t_r^2} \\ &= \frac{\lambda_1 t_r + \lambda_3 k_g}{k_g^2 + t_r^2} \end{aligned} \tag{65}$$

The ruled surface is developable if and only if $P_{\overline{Y}} = 0$ and also

$$\frac{\lambda_1}{\lambda_3} = -\frac{k_g}{t_r}$$

Here if $k_g = 0$, then the curve β is geodesic curve. Subsequently, if $\widetilde{\Phi}_{\overline{N}} = \beta + vN$ then we get

$$\begin{aligned} P_{\overline{N}} &= \frac{\det(\beta', N, N')}{\|N'\|^2} \\ &= \frac{\det(\lambda_1 T + \lambda_2 Y + \lambda_3 N, N, -k_n T - t_r Y)}{k_n^2 + t_r^2} \\ &= \frac{\lambda_1 t_r}{k_n^2 + t_r^2}. \end{aligned} \tag{66}$$

As stated above smiliarly, the ruled surface that corresponds to $\overline{N}$ is developable if and only if $\lambda_1 t_r = 0$. And if $t_r = 0$, then the curve β is a curvature line.

$$\begin{bmatrix} \overline{T}' \\ \overline{Y}' \\ \overline{N}' \end{bmatrix} = \begin{bmatrix} 0 & k_g + \varepsilon\lambda_3 & k_n - \varepsilon\lambda_2 \\ -k_g - \varepsilon\lambda_3 & 0 & t_r + \varepsilon\lambda_1 \\ -k_n + \varepsilon\lambda_2 & -t_r - \varepsilon\lambda_1 & 0 \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{Y} \\ \overline{N} \end{bmatrix} \tag{67}$$

According to this, the Darboux vector of the motion is:

$$\overline{W} = (t_r + \varepsilon\lambda_1)\overline{T} + (-k_n + \varepsilon\lambda_2)\overline{Y} + (k_g + \varepsilon\lambda_3)\overline{N} \tag{68}$$

Subsequently,

$$\begin{aligned}\left\|\overline{W}\right\| &= \sqrt{(t_r+\varepsilon\lambda_1)^2+(-k_n+\varepsilon\lambda_2)^2+(k_g+\varepsilon\lambda_3)^2} \\ &= \sqrt{(t_r^2+k_n^2+k_g^2)+2\varepsilon(\lambda_1 t_r-\lambda_2 k_n+\lambda_3 k_g)} \\ &= \sqrt{(t_r^2+k_n^2+k_g^2)}+\varepsilon\frac{(\lambda_1 t_r-\lambda_2 k_n+\lambda_3 k_g)}{\sqrt{(t_r^2+k_n^2+k_g^2)}} \\ &= \overline{\Psi}+\varepsilon\overline{\Psi}^*\end{aligned} \tag{69}$$

This is the dual rotation Pfaffian vector. The real part $\overline{\Psi}$ and dual part $\overline{\Psi}^*$ correspond to the rotation and translation motions. In order to leave out the pure translation we will suppose that $\overline{\Psi}\neq 0$. According to Pfaffian vector, it can be easily said that if $\overline{\Psi}\neq 0, \overline{\Psi}^*\neq 0$, then there are rotation and translation motions (helix motion).

And also if $\overline{\Psi}\neq 0, \overline{\Psi}^*=0$ it can be obtained pure rotation (spherical motion).

On the other hand, the pitch of this motion can be given by

$$\begin{aligned}\frac{\overline{\Psi}^*}{\overline{\Psi}} &= \frac{d\frac{(\lambda_1 t_r-\lambda_2 k_n+\lambda_3 k_g)}{\sqrt{(t_r^2+k_n^2+k_g^2)}}}{\sqrt{(t_r^2+k_n^2+k_g^2)}} \\ &= \frac{(\lambda_1 t_r-\lambda_2 k_n+\lambda_3 k_g)}{(t_r^2+k_n^2+k_g^2)}\end{aligned} \tag{70}$$

Here, taking the curvature of the curve β as

$$k_n^2+k_g^2=\overline{\kappa}^2$$

the pitch of the instantenous screw motion can be given by:

$$\begin{aligned}\frac{\overline{\Psi}^*}{\overline{\Psi}} &= \frac{d\frac{t_r}{\sqrt{t_r^2+\overline{\kappa}^2}}}{\sqrt{t_r^2+\overline{\kappa}^2}} \\ &= \frac{t_r}{t_r^2+\overline{\kappa}^2}\end{aligned} \tag{71}$$

Result. When the curve β isn't a curve line, we have the instantaneous screw axis where the pitch is

$$\frac{\overline{\Psi}^*}{\overline{\Psi}}=\frac{(\lambda_1 t_r-\lambda_2 k_n+\lambda_3 k_g)}{(t_r^2+k_n^2+k_g^2)} \tag{72}$$

Result. When the curve β is a curvature line that is to say $t_r=0$, we have $\overline{\Psi}\neq 0, \overline{\Psi}^*=0$ and it can be obtained pure rotation

$$\frac{\overline{\Psi}^*}{\overline{\Psi}}=\frac{-\lambda_2 k_n+\lambda_3 k_g}{k_n^2+k_g^2} \tag{73}$$

Result. If the curve β is geodesic, then

$$\frac{\overline{\Psi}^*}{\overline{\Psi}} = \frac{(\lambda_1 t_r - \lambda_2 k_n)}{(t_r^2 + k_n^2)} \tag{74}$$

Result. If the curve β is asymptotic then the pitch is

$$\frac{\overline{\Psi}^*}{\overline{\Psi}} = \frac{\lambda_1 t_r + \lambda_3 k_g}{t_r^2 + k_g^2}.$$

4 CONCLUSIONS

The starting point of this study is to define a different motion with two unit dual spheres moving with respect to each aother. Using the same way, as in [11], we develope this approach. According to this approach, the motion is investigated in two different ways, by taking two different dual curves with their orthonormal frames. After giving the relation between dual motion and real spatial motion, the developability of the ruled surfaces that correspond to these curves are presented. The pitch and Darboux vectors of the motions are obtained. Moreover, some results and examples are given for geodesic curves and curvature lines.

We hope that this study will gain different interpretation to the other studies in this field.

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Received: December, 2011