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Frame Characterization of Hilbert-Schmidt Operator

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Abstract. In this paper we prove that the Hilbert-Schmidt operator norm $\|T\|_{HS}$ can be defined using frames. For this, we first prove some lemmas to show the norm is independent of the choice of frames.

1. Introduction

A lot of problems in physics and other application areas can be formulated as operator theory such as differential and integral equations. An interesting class of operators is the Hilbert-Schmidt class [3]. Peter Balazs [4] has discussed some related topic. In this paper we use different method to characterize Hilbert-Schmidt operator using frame theory. We will denote infinite dimensional Hilbert spaces by H and their inner product with $< .,. >$ which is linear in the first coordinate. A bounded operator $T : H \to H$ is called a Hilbert-Schmidt operator if there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H such that $\sum_{i=1}^{\infty} \|T(e_i)\|^2 < \infty$.

Let T^* be the adjoint operator of T and $\langle , \rangle$ be the usual inner product. If $\{\xi_i\}_{i=1}^{\infty}$ is another orthonormal basis for H such that $\sum_k \sum_j |\langle T\xi_k, e_j \rangle|^2 < \infty$, then we have

$$\begin{aligned}\sum_{i=1}^{\infty} \|T(e_i)\|^2 &= \sum_{i=1}^{\infty} \|T(\xi_i)\|^2 = \sum_k \sum_j |\langle T\xi_k, e_j \rangle|^2 \\ &= \sum_k \sum_j |\langle T^* e_j, \xi_k \rangle|^2 = \sum_j \|T^* e_j\|^2.\end{aligned}$$

Thus Hilbert-Schmidt norm of T is given by $\|T\|_{HS} = \sqrt{\sum_{i=1}^{\infty} \|T(e_i)\|^2}$.

Definition 1.1. [1, 2] A sequence $\{f_i\} \subset H$ is called a frame for the (separable) Hilbert space H, if there exist constants $A, B > 0$, such that

$$A\|f\|_H^2 \leq \sum_i | < f, f_i > |^2 \leq B\|f\|_H^2, \quad \text{for all } f \in H$$

A and B are called frame lower and upper bounds respectively

2. MAIN RESULTS

In this section we prove that the Hilbert-Schmidt operator norm $\|T\|_{HS}$ can be defined using frames. For this, we prove first some lemmas to show the norm is independent of the choice of frames.

Lemma 2.1. *Let $T : H \to H$ be a bounded operator, $\{f_i\}_{i=1}^{\infty}$ be a frame for H with lower and upper bounds A and B, respectively. If $\sum_{i=1}^{\infty} \|T(f_i)\|^2 < \infty$, then $\sum_{i=1}^{\infty} \|T^*(f_i)\|^2 < \infty$ and*

$$\frac{A}{B}\sum_{i=1}^{\infty} \|T(f_i)\|^2 \leq \sum_{i=1}^{\infty} \|T^*(f_i)\|^2 \leq \frac{B}{A}\sum_{i=1}^{\infty} \|T(f_i)\|^2$$

Proof. Note that for all $i \in N$,

$$\|T^*(f_i)\|^2 \leq \frac{1}{A}\sum_{j=1}^{\infty} |\langle T^*(f_i), f_j\rangle|^2 = \frac{1}{A}\sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2$$

$$\begin{aligned}\sum_{i=1}^{\infty} \|T^*(f_i)\|^2 &\leq \frac{1}{A}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2 = \frac{1}{A}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\langle T(f_j), f_i\rangle|^2 \\ &\leq \frac{B}{A}\sum_{j=1}^{\infty} \|T(f_j)\|^2 < \infty\end{aligned}$$

Thus,

$$\sum_{i=1}^{\infty} \|T^*(f_i)\|^2 \leq \frac{B}{A}\sum_{j=1}^{\infty} \|T(f_j)\|^2.$$

On the other hand, for all $i \in N$,

$$\|T^*(f_i)\|^2 \geq \frac{1}{B}\sum_{j=1}^{\infty} |\langle T^*(f_i), f_j\rangle|^2 = \frac{1}{B}\sum_{j=1}^{\infty} |\langle T(f_j), f_i\rangle|^2$$

and hence,

$$\begin{aligned}\sum_{i=1}^{\infty} \|T^*(f_i)\|^2 &\geq \frac{1}{B}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle T(f_j), f_i\rangle|^2 \\ &= \frac{1}{B}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\langle T(f_j), f_i\rangle|^2 \geq \frac{A}{B}\sum_{j=1}^{\infty} \|T(f_j)\|^2 < \infty\end{aligned}$$

Therefore,

$$\frac{A}{B}\sum_{j=1}^{\infty} \|T(f_j)\|^2 \leq \sum_{i=1}^{\infty} \|T^*(f_i)\|^2 \leq \frac{B}{A}\sum_{j=1}^{\infty} \|T(f_j)\|^2.$$

□

Lemma 2.2. *Let $T : H \to H$ be a bounded operator, and let $\{f_i\}_{i=1}^{\infty}$, $\{g_i\}_{i=1}^{\infty}$ be frames for H with lower and upper bounds A_1, B_1 and A_2, B_2, respectively. If $\sum_{i=1}^{\infty} \|T(f_i)\|^2 < \infty$, then $\sum_{i=1}^{\infty} \|T(g_i)\|^2 < \infty$ and*

$$\frac{A_1A_2}{B_1^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2 \leq \sum_{i=1}^{\infty} \|T(g_i)\|^2 \leq \frac{B_1B_2}{A_1^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2$$

Proof. Note that for all $i \in N$,

$$\|T(g_i)\|^2 \leq \frac{1}{A_1} \sum_{j=1}^{\infty} |\langle T(g_i), f_j\rangle|^2 = \frac{1}{A_1} \sum_{j=1}^{\infty} |\langle T^*(f_j), g_i\rangle|^2$$

and

$$\|T(g_i)\|^2 \geq \frac{1}{BA_1} \sum_{j=1}^{\infty} |\langle T^*(f_j), g_i\rangle|^2.$$

Thus,

$$\begin{aligned} \sum_{i=1}^{\infty} \|T(g_i)\|^2 &\leq \frac{1}{A_1} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle T^*(f_j), g_i)\rangle|^2 = \frac{1}{A_1} \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\langle T^*(f_j), g_i\rangle|^2 \\ &\leq \frac{B_2}{A_1} \sum_{j=1}^{\infty} \|T^*(f_j)\|^2 < \infty \end{aligned}$$

and similarly,

$$\sum_{i=1}^{\infty} \|T(g_i)\|^2 \geq \frac{A_2}{B_1} \sum_{j=1}^{\infty} \|T^*(f_j)\|^2.$$

Thus,

$$\frac{A_2}{B_1} \sum_{i=1}^{\infty} \|T^*(f_i)\|^2 \leq \sum_{i=1}^{\infty} \|T(g_i)\|^2 \leq \frac{B_2}{A_1} \sum_{i=1}^{\infty} \|T^*(f_i)\|^2.$$

By Lemma 2.1, we get

$$\frac{A_1A_2}{B_1^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2 \leq \sum_{i=1}^{\infty} \|T(g_i)\|^2 \leq \frac{B_1B_2}{A_1^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2.$$

□

Corollary 2.3. *Let $T : H \to H$ be a bounded operator, and let $\{f_i\}_{i=1}^{\infty}$ be a frame for H with lower and upper bounds A, B, respectively, with $\sum_{i=1}^{\infty} \|T(f_i)\|^2 < \infty$, then for any orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H, we have*

$$\frac{A}{B^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2 \leq \sum_{i=1}^{\infty} \|T(e_i)\|^2 \leq \frac{B}{A^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2,$$

and hence,

$$\frac{A^2}{B} \sum_{i=1}^{\infty} \|T(e_i)\|^2 \leq \sum_{i=1}^{\infty} \|T(f_i)\|^2 \leq \frac{B^2}{A} \sum_{i=1}^{\infty} \|T(e_i)\|^2.$$

Corollary 2.4. *Let $T : H \to H$ be a bounded operator. Then T is a Hilbert-Schmidt operator if and only if there exists a frame $\{f_i\}_{i=1}^{\infty}$ for H such that $\sum_{i=1}^{\infty} \|T(f_i)\|^2 < \infty$.*

In this case,

$$\frac{A}{B^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2 \leq \|H\|_{HS}^2 \leq \frac{B}{A^2} \sum_{i=1}^{\infty} \|T(f_i)\|^2,$$

where A, B are the lower and upper frame bounds respectively.

Proof. If T is a Hilbert-Schmidt operator, then there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H such that $\sum_{i=1}^{\infty} \|T(e_i)\|^2 < \infty$. Conversely, if $\sum_{i=1}^{\infty} \|T(f_i)\|^2 < \infty$ for some frame $\{f_i\}_{i=1}^{\infty}$ for H, then we get the result by Lemma 2.2 and Corollary 2.3. □

Corollary 2.5. *If $T : H \to H$ is a Hilbert-Schmidt operator then*

$$\|T\|_{HS} = \sqrt{\sum_{i=1}^{\infty} \|T(f_i)\|^2} < \infty$$

for any tight frame $\{f_i\}$ for H.

Lemma 2.6. *Let $T : H \to H$ is a Hilbert-Schmidt operator and let $\{f_i\}$ be a frame for H with lower and upper bounds A and B respectively, then*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2 < \infty$$

and

$$\frac{A^4}{B^2} \|T\|_{HS}^2 \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2 \leq \frac{B^4}{A^2} \|T\|_{HS}^2.$$

Proof. Note that,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2 \quad \leq \quad \frac{B^2}{A} \sum_{i=1}^{\infty} | \ T(f_i)\|^2 \leq \infty.$$

Similarly,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2 \geq \frac{A^2}{B} \sum_{i=1}^{\infty} | \ T(f_i)\|^2.$$

By Corollary 2.4, we get that

$$\frac{A^4}{B^2} \|T\|_{HS}^2 \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle f_i, T(f_j)\rangle|^2 \leq \frac{B^4}{A^2} \|T\|_{HS}^2.$$

□

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