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Singular Perturbation with a Reduced Approximation Order in Space for the Transport Equation

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Abstract

This work is devoted to singular transport phenomena by convection with fast-low time scales, or to transport in porous media with vanishing discontinuous porosities. For $P^1 - P^0$ finite element, by using a reduction of the approximation order for the time differential operator, we propose a numerical method which does not have any oscillations in the neighborhood of the coefficient discontinuity. Error estimates of order one with respect to space are provided. Euler explicit and implicit time schemes are proposed, and by considering a toy problem, the order one of convergence with respect to time and space is checked.

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1 Introduction

A reduced order approximation in space for the time operator is proposed when the transport equation is singularly perturbed (i.e. has discontinuous vanishing coefficients). The presented method is inspired from the singular dynamics introduced in [7]. For the sake of clarity, a 1-D case is considered with one interval where the coefficient vanishes. The method can be generalized for higher space dimensions and for coefficients vanishing on many parts of the space domain. This projection method, is designed for considering some problems with fast-low dynamics in population dynamics [1] ; some plants biology problems [8] or some problems in neurobiology [6]. Some dissolution problems with acid in porous media with discontinuous porosity, lead to singular perturbation of the transport equation with discontinuous coefficients for the time

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operator [5]. It is worth to notice that the numerical solution of the transport equation with vanishing discontinuous coefficients usually presents some local oscillations in a neighborhood of discontinuities of coefficients (see for example the results with a least squares method in [4] p. 115) which is not the case for the proposed method. To the knowledge of the authors, there does not exist proof of error estimates in the literature for a projection method in the context of Finite Element with a reduced order when the differential operator degenerates.

Let function

$$f = \begin{cases} f_1(x,t) & 0 < x < \frac{1}{2}, \\ f_2(x) & \frac{1}{2} < x < 1, \end{cases} \tag{1}$$

and u_0 be given, the domain is defined by $\Omega = (0,1) \times (0,1)$, and we consider the following problem: find u solution to:

$$\begin{cases} 1_{[0,\frac{1}{2}]}(x)\partial_t u(x,t) + \partial_x u(x,t) + u(x,t) = f(x,t); \text{ in } \Omega, \\ u(x,0) = u_0 \qquad 0 < x < \frac{1}{2}; \quad u(0,t) = 0 \quad 0 < t < 1. \end{cases} \tag{2}$$

2 Existence of solutions

We denote by $\partial\Omega_- = \{(x,t) = \{0\} \times (0,1) \cup (0,1/2) \times \{0\}\}$ the incoming part of the boundary of Ω, then the $L^2(\Omega)$-unbounded degenerated differential operator A is defined by:

$$Au = 1_{[0,\frac{1}{2}]}(x)\partial_t u + \partial_x u + u.$$

The graph-norm of the operator A is given by:

$$|||\varphi|||^2 = \|A\varphi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)}.$$

The space in which the solution is searched is defined by the following closure: $E = \overline{\{\varphi \in \mathcal{D}(\overline{\Omega}),\ \varphi_{|\partial\Omega_-} = 0.\}}^{|||\cdot|||}$, where $\mathcal{D}(\overline{\Omega})$ is the set of $C^\infty(\overline{\Omega})$ functions compactly supported. Taking into account the estimate

$$\int_\Omega Avv\,dt\,dx \geq \|v\|^2_{L^2(\Omega)},$$

we have that the operator A is linear continuous with a closed range. For numerical purposes, we need to introduce a variational formulation of the problem.

Let the Sobolev's space $H^1(0,1) = \{\varphi \in l^2(0,1);\ \frac{d\varphi}{dx} \in l^2(0,1)\}$ the subspace of function belonging to $H^1(0,1)$ with value zero at $x = 0$ is denoted by V. Let the bilinear forms $a(\cdot,\cdot)$ defined from $V \times L^2(0,1)$ and $b(\cdot,\cdot)$ defined from $V \times L^2(0,1/2)$ be such that:

$$a(v,w) = \int_0^1 \partial_x v(x)w(x) + v(x)w(x)dx; \quad b(v,\varphi) = \int_0^{\frac{1}{2}} w(x)\varphi(x)\,dx. \tag{3}$$

In what follows, the restriction operator is defined.

Lemma 2.1 *The operator $B : V \to L^2(0,1/2)$ associated to the bilinear form $b(\cdot,\cdot)$ is the linear continuous restriction operator to the segment $[0,1/2]$. We have $\ker B = {}_0H^1(1/2,1)$ and the range of its adjoint is given by Ra $B^* = {}_0H^1(0,1/2)$ where the left subscript 0 indicates that only functions vanishing at the left end-point of the interval are considered.*

Let introduce a variational formulation for the problem (2):

$$\left\{\begin{array}{l} \text{find } u(t) \in C^0([0,1],V); u(0) = u_0 \text{ satisfying} \\ < B\dot{u}(t), Bw >_{L^2(0,\frac{1}{2}),L^2(0,\frac{1}{2})} + a(u(t),w) = (f(t),w), \quad \forall w \in L^2(0,\frac{1}{2}), \end{array}\right. \tag{4}$$

where $\dot{u}$ is the derivative of u with respect to time. If the space V is decomposed as $V = \ker B \oplus \ker B^{\perp} = \ker B \oplus$ Ra B^*. The bilinear form $a(\cdot,\cdot)$ verifies the inf-sup conditions thus it is deduced that the problem (4) is well-posed.

3 Semi-discretized finite element formulation for the problem (4)

Let $M = 2m$ for a given integer m, set $h = \frac{1}{M}$, and the interval $[0,1]$ is split in sub-intervals $I_i = (x_{i-1}, x_i)$ with $x_i = ih$ for $1 \leq i \leq M$. Let $V_h \subset \{\varphi \in H^1(0,1); \varphi(0) = 0\}$ be the sub-space generated by the M hat Lagrange's functions of order one φ_i, equipped with the H^1-semi-norm. Let $H_h \subset L^2(0,1)$ be the sub-space generated by the M constant functions ψ_i on each sub-interval I_i, and let $M_h \subset L^2(0,\frac{1}{2})$ be the sub-space generated by q_i the $\frac{M}{2}$ constant functions on each sub-intervals $I_i \subset (0,\frac{1}{2})$. Let $P_{V_h} : C^0[0,1] \to V_h$ the Lagrange's interpolate operator and let $\Pi_h : L^2(0,1) \to M_h$ and $P_{H_h} : L^2(0,1) \to H_h$ be two L^2-projectors. Finally introduce the following matrices :

$$\left\{\begin{array}{l} A^-_{ij} = \int_0^{\frac{1}{2}} \partial_x\varphi_j(x)\psi_i + \varphi_j(x)\psi_i(x)dx, \\ A^+_{ij} = \int_{\frac{1}{2}}^1 \partial_x\varphi_j(x)\psi_i + \varphi_j(x)\psi_i(x)dx, \ 1 \leq i,j \leq M; \\ B_{ij} = \int_0^1 \varphi_j(x)q_i(x)dx, 1 \leq i \leq m; 1 \leq j \leq M; \\ C_{ij} = \int_0^{\frac{1}{2}} q_j(x)q_i(x)dx; 1 \leq i,j \leq m. \end{array}\right. \tag{5}$$

Definition 3.1 *For $u_0 \in V_h$ and $F(t) = P_{H_h}f(t)$ given, by denoting by $A = A^- + A^+$, $u_h(t) \in V_h; v_h(t) \in M_h$ the solution of the semi-discretized problem verifies:*

$$\left\{\begin{array}{l} B^t v_h(t) + Au_h(t) = F(t), \\ Cv_h(t) = B\dot{u}_h(t), \qquad \forall t \in]0,1]. \\ u_h(0) = u_0, \end{array}\right. \tag{6}$$

Here $u_h(t)$ and $v_h(t)$ stand for functions of V_h or corresponding vectors of degrees of freedom.

Theorem 3.2 *For $u_0 = 0$, $f \in C^1(\Omega)$ given, and assuming that $u \in C^1([0,1]; H^2(0,1))$, there exists $C(f)$ such that for all $h > 0$, the following error estimate holds true:*

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \leq C(f)h \quad 0 < t \leq 1. \tag{7}$$

For the proof we need of some a priori estimates.

Lemma 3.3 *There exist C_1; C_2 such that for all $h > 0$ the following estimates hold true:*

$$\begin{aligned} &C_1\|u_h\|^2_{L^\infty(0,1;V_h)} + \|\Pi_h P_{H_h}\dot{u}_h\|^2_{L^2(\Omega)} + \|P_{H_h}u_h\|^2_{L^\infty(0,1;L^2(0,1))} \leq \\ &C_2\Big(\|\dot{f}\|^2_{L^2(\Omega)} + \|f\|^2_{L^\infty(0,1;L^2(0,1))}\Big); \\ &\|(I - \Pi_h)\dot{u}_h\|^2_{L^2(0,1;L^2(0,1))} = 0. \end{aligned} \tag{8}$$

Please note that $\Pi_h P_{H_h} = \Pi_h$. The semi-discretized problem becomes:

$$(\dot{u}_h(t), \Pi_h w_h) + a(u_h(t), w_h) = (f(t), w_h), \quad \forall w_h \in H_h. \tag{9}$$

Successively choose $w_h = \partial_x \dot{u}_h(t) \in H_h$, and $w_h = P_{H_h}\dot{u}_h(t) \in H_h$, adding the two equations and integrate between 0 and t. The term of the first equation $\int_0^t \int_0^1 u_h(t)\partial_x\dot{u}_h(t)\,dx\,dt$ is integrated by parts, the following term

$$-\int_0^t \int_0^1 \partial_x u_h(t)\dot{u}_h(t)\,dx\,dt$$

will be compensated with the term

$$\int_0^t \int_0^1 \partial_x u_h(t) P_{H_h}\dot{u}_h(t)\,dx\,dt$$

of the second equation since we have $\partial_x u_h(t) \in H_h$. So the first estimate is deduced. For the second, choose $w_h = (I - \Pi_h)\psi_h$ for $\psi_h \in H_h$. We have:

$$a(u_h(t), (I - \Pi_h)\psi_h) = (f(t), (I - \Pi_h)\psi_h).$$

Take the derivative with respect to t, since f does not depend on time for $x > \frac{1}{2}$, the second equality is obtained.
Let us give a sketch of the proof of the theorem 3.2. Let us denote the error by $e_h(t) = u_h(t) - P_{V_h}u(t)$. The error equation is given by: $\forall w_h \in H_h$

$$\begin{aligned} &(\dot{e}_h(t), w_h) + a(\dot{e}_h(t), w_h) = \\ &a((I - P_{V_h})u(t), w_h) + (\dot{u}_h, (I - \Pi_h)w_h) + ((I - P_{V_h})\dot{u}(t), w_h). \end{aligned} \tag{10}$$

Choose $w_h = P_{H_h} e_h(t)$, integrate the equation (10) between 0 and t and writes this expression as I=II+III+IV. We have:

$$\begin{array}{l} \|P_{H_h} e_h\|^2_{L^\infty(0,1;L^2(0,1))} \leq I \quad (\text{ since } \quad \partial_x e_h \in H_h); \\ |II| \leq \|\partial_x (I - P_{V_h}) u\|_{L^2(\Omega)} \|P_{H_h} e_h\|_{L^2(\Omega)}; \\ |III| \leq \|(I - \Pi_h) \dot{u}_h\|_{L^2(\Omega)} \|P_{H_h} e_h\|_{L^2(\Omega)}; \\ |IV| \leq \|(I - P_{H_h}) \dot{u}\|_{L^2(\Omega)} \|P_{H_h} e_h\|_{L^2(\Omega)}. \end{array} \tag{11}$$

The classical stability results and estimates for $(I - P_\bullet)$ as function of h for P_{V_h} and for P_{H_h} and Π_h (see [3] for example) lead to the conclusion since u is enough regular and since the a priori estimates of Lemma 3.3 hold true. Expressing

$$u - u_h = (I - P_{H_h})u + P_{H_h}(I - P_{V_h})u + P_{H_h} e_h + (P_{H_h} - I)u_h,$$

we get the error estimate.

4 Implicit and Explicit Euler's schemes in time

In the problem (6), eliminate the unknown v, and for K fixed define $\Delta t = \frac{1}{K}$ and the times sequence $t_k = k\Delta t$; $0 \leq k \leq K$. For u_h^0 given, the implicit Euler's scheme reads:

$$B^t C^{-1} B u_h^{k+1} + \Delta t A u_h^{k+1} = \Delta t F(t_{k+1}) + B^t C^{-1} B u_h^k. \tag{12}$$

Please remark that $B^t C^{-1} B + \Delta t A$ is invertible since it is the sum of a semi-definite matrix and of a positive definite matrix. The explicit Euler's scheme reads:

$$B^t C^{-1} B u_h^{k+1} + A^+ u_h^{k+1} = \Delta t F^-(t_k) + B^t C^{-1} B u_h^k - \Delta t A^- u_h^k + F^+(t_{k+1}). \tag{13}$$

with $F^-(t) = P_{H_h} 1_{[0,\frac{1}{2}]}(x) f(t,x)$; $F^+(t) = P_{H_h} 1_{[\frac{1}{2},1]}(x) f(t,x)$.

Let us end this note with two convergence curves for the problem (2), when the right end side is given by: $f = \begin{cases} (x - 1/2)^2 t^2 + x^2 & 0 < x < 1/2 \\ x^2 \quad 1/2 < x < 1 \end{cases}$; and $u_0(x) = x^2$, $0 < x < 1/2$. In figure 1, L^2 error curves in logarithmic scale for the scheme (13) are presented. These results agree with the announced results. To conclude, please remark that an upwind finite element in space could have been used as in [2].

The presented results can be generalized to diffusion problems with two time scales. Explicit or implicit Euler's schemes can be obtained.

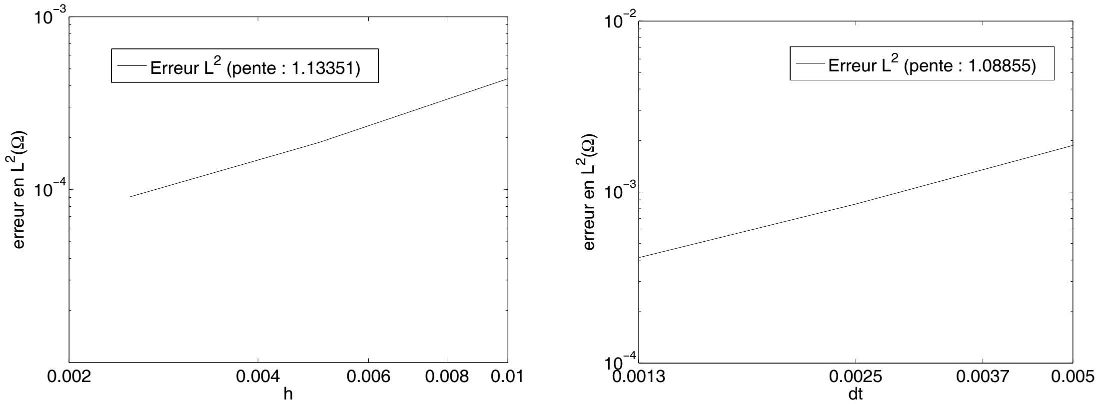

Figure 1: Space error (left), time error (right).

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