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A Note on D. Bartl's Algebraic Proof of Farkas's Lemma

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Abstract

Following Gale, we give a down-to-earth proof of a version of Farkas's Lemma proposed by Bartl, which is valid for vector spaces over any linearly ordered field, not necessarily commutative. In passing, we use the same technique to prove Gordan's Theorem in the analogous generalized form. The close relation between Farkas's Lemma and Gordan's Theorem opens a door for possible generalization of Bartl's formulation to other related theorems, which we discuss briefly in the conclusion.

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1 Introduction

According to Broyden ([3]), the first correct proof of Farkas's Lemma was published (in Hungarian) in 1898, but Farkas's best-known exposition of his famous lemma appeared (in German) in 1902 ([4]). Since its appearance, it has been studied extensively. Gordan's Theorem ([6]) predated Farkas's Lemma, but seemed to receive much less attention. We briefly remark that Farkas's Lemma is a useful tool for people working in optimization, game theory and linear programming, just to name a few. For more precise descriptions and more detailed history, the reader is referred to, for example, the papers by Broyden ([3]) and by Bartl ([2]). In this note, we will give a down-to-earth proof of the version of Farkas's Lemma proposed by Bartl in his paper ([2]), closely following Gale's proof ([5], p.44). This gives some simplification in the sense that the current proof is more conceptual, and the algebra involved is totally down-to-earth. In passing, we also prove Gordan's Theorem, using similar technique. Note that in the classical setting, with the ground field replaced by any linearly ordered field, we know that many key results such as Farkas's Lemma, variant of Farkas's Lemma ([1]), some versions of separating hyperplane theorems (pertaining to linear polyhedral cones), Gale's Theorem (see [7]), and Gordan's Theorem ([6]) (this list can be enlarged to include most of the theorems of alternatives discussed in [7]) are actually equivalent (see [8]). This observation opens the door for future investigation of generalizations for the related theorems.

2 Notations

F: a linearly ordered field, not necessarily commutative.
W: a right vector space over F, whose dimension may be finite or infinite.
V: a right vector space over F.
α_i: linear functional on W, i.e. $\alpha_i : W \to F$ is a linear map. We endow the space of linear functionals with a left vector space structure. We make this choice so that the expression $a\alpha_i(xb) = a\alpha_i(x)b$ makes natural sense, where $a, b \in F$, and $x \in W$.
γ: a linear transformation from W to V, compatible with the vector space structures of W and V, namely $\gamma(xb) = \gamma(x)b$, for all $x \in W$ and all $b \in F$, etc.
The symbol A stands for the linear transformation $A : W \to F^m$. A linear transformation A is equivalent to the information of m linear functionals, $\alpha_i : W \to F$, $1 \leq i \leq m$, where $\alpha_i(x)$ represents the i-th component of the column vector Ax with $x \in W$.
We denote the ordering of F and V by the symbols "$\leq$" and "$\preceq$", respectively. For the ordering, the following five statements must hold true for $u, v \in V$ and any $\mu \in F$ so that V is a linearly ordered vector space over the linearly ordered field F:

i) $u \preceq v$ if and only if $u - v \preceq 0$
ii) $u \succeq 0$ or $u \preceq 0$
iii) if $u \succeq 0$ and $u \preceq 0$, then $u = 0$
iv) if $u \succeq 0$ and $v \succeq 0$, then $u + v \succeq 0$
v) if $\mu \geq 0$ and $u \succeq 0$, then $u\mu \succeq 0$

3 Main Results

We will prove the following generalized form of Farkas's Lemma, which is Lemma 2 of [2].

Farkas's Lemma 3.1 *Let* $A : W \to F^m$ *and* $\gamma : W \to V$ *be linear mappings. Then either (A) there exists an* $x \in W$ *such that* $\alpha_1(x) \geq 0, \cdots, \alpha_m(x) \geq 0$, *and* $\gamma(x) \prec 0$, *or (B) there exist nonnegative vectors* $u_1, \cdots, u_m \in V$ *such that* $\gamma = u_1\alpha_1 + \cdots + u_m\alpha_m$. *The alternatives (A) and (B) exclude each other.*

Proof. First of all, for any m, it is clear that (A) and (B) cannot be both true, for this would imply that $\gamma(x) = u_1\alpha_1(x) + \cdots + u_m\alpha_m(x) \succeq 0$ contradicting the condition $\gamma(x) \prec 0$.

For the remaining part, we will assume that (B) is false, and proceed to establish (A). We prove this by induction on m.

For $m = 1$, that (B) is false implies that $\gamma \neq 0$, otherwise letting $u_1 = 0$ would give $\gamma = u_1\alpha_1$, contrary to the assumption. Therefore there exists $x_1 \in W$ such that $\gamma(x_1) \prec 0$.

Now if $\alpha_1(x_1) \geq 0$, we would be finished. Therefore assume that $\alpha_1(x_1) < 0$.

We define $\bar{\gamma}$ by $\bar{\gamma} = \gamma - \gamma(x_1)\alpha_1(x_1)^{-1}\alpha_1$. Necessarily $\bar{\gamma} \neq 0$, otherwise γ would satisfy (B). Since $\bar{\gamma} \neq 0$, there exists $\bar{x} \in W$ such that $\bar{\gamma}(\bar{x}) \prec 0$.

Letting now $x = \bar{x} - x_1\alpha_1(x_1)^{-1}\alpha_1(\bar{x})$, it is straightforward to check that $\gamma(x) = \bar{\gamma}(\bar{x}) \prec 0$, and $\alpha_1(x) = 0$, hence the case of $m = 1$ is proven.

We assume that the case $m - 1 \geq 1$ is true, and prove for the case m. Assume (B) is false, i.e. there do not exist $u_i \succeq 0$ such that $\sum_{i=1}^{m} u_i\alpha_i = \gamma$, which also implies that there do not exist $u_i \succeq 0$ such that $\sum_{i<m} u_i\alpha_i = \gamma$. By the induction hypothesis, there exists $x_1 \in W$ such that $\alpha_i(x_1) \geq 0$ for all $i < m$ and $\gamma(x_1) \prec 0$. If $\alpha_m(x_1) \geq 0$, we would be finished, hence we may assume that $\alpha_m(x_1) < 0$.

Now define $\bar{\alpha_i}$ by

$$\bar{\alpha_i} = \alpha_i - \alpha_i(x_1)\alpha_m(x_1)^{-1}\alpha_m, \forall i < m \tag{1}$$

and define $\bar{\gamma}$ by

$$\bar{\gamma} = \gamma - \gamma(x_1)\alpha_m(x_1)^{-1}\alpha_m. \tag{2}$$

It is easy to see that there do not exist $\bar{u_i} \succeq 0$ such that

$$\sum_{i<m} \bar{u_i}\bar{\alpha_i} = \bar{\gamma}. \tag{3}$$

Otherwise, substituting (1) and (2) into (3) would yield the following equation

$$\sum_{i<m} \bar{u_i}\alpha_i + \left(\gamma(x_1)\alpha_m(x_1)^{-1} - \sum_{i<m} \bar{u_i}\alpha_i(x_1)\alpha_m(x_1)^{-1}\right)\alpha_m = \gamma,$$

where the coefficient vectors for the α_i's, $1 \leq i \leq m$, are all nonnegative, a contradiction.

By the induction hypothesis again, applied this time to $\bar{\alpha_i}, i < m$ and $\bar{\gamma}$, there exists $\bar{x} \in W$ such that $\bar{\alpha_i}(\bar{x}) \geq 0, i < m$ and $\bar{\gamma}(\bar{x}) \prec 0$.

Letting $x = \bar{x} - x_1\alpha_m(x_1)^{-1}\alpha_m(\bar{x})$, it is straightforward to check that

$$\alpha_i(x) = \bar{\alpha_i}(\bar{x}) \geq 0, \forall i < m, \alpha_m(x) = 0, \text{ and } \gamma(x) = \bar{\gamma}(\bar{x}) \prec 0,$$

therefore we are done.

□

Remarks.

1. In [2], the vector spaces W, V and F^m are assumed to be left vector spaces over F, but the author only mentioned this for F^m. To handle the coefficients for the vectors u_i, being vectors from left vector space over F, the author used the notation $\iota u_i\alpha_i$ to denote the function from W to V, satisfying $\iota u_i\alpha_i(x) = \alpha_i(x)u_i$, while our notation is down-to-earth, where we simply prescribe that V is a right vector space and therefore $u\alpha$ as a function from W to V makes natural sense, namely $u\alpha(x) = u \cdot \alpha(x)$. We can also prove the result using the assumption that W, V, and F^m are all left vector spaces; in this situation the set of linear functionals from W to F naturally form a right vector space over F, which is why the author of [2] defined the operation $\iota\lambda\alpha$ for $\lambda \in F$, which is a functional from W to F satisfying $\iota\lambda\alpha(x) = \alpha(x) \cdot \lambda$. We point out here only the key modifications needed:

$$\bar{\alpha_i} = \alpha_i - \alpha_m \cdot \alpha_m^{-1}(x_1)\alpha_i(x_1)$$

and

$$\bar{\gamma} = \gamma - \alpha_m \cdot \alpha_m^{-1}(x_1)\gamma(x_1).$$

Of course this is essentially what the author of [2] had, namely

$$\bar{\alpha_i} = \alpha_i - \iota\lambda_i\alpha_m$$

and

$$\bar{\gamma} = \gamma - \iota v\alpha_m,$$

where $\lambda_i = \alpha_m^{-1}(x_1)\alpha_i(x_1)$ and $v = \alpha_m^{-1}(x_1)\gamma(x_1)$.

2. We note here that the result of the generalized Farkas's Lemma above can be reduced to the case when W is a finite dimensional vector space easily. By linear algebra, $W = W' \oplus \ker A$ is a direct sum of W' and the kernel of A, where $W' \simeq \mathrm{im}A$ is a finite dimensional vector space. Assume that condition (B) in the Farkas's Lemma is false, we proceed to establish (A). If $\gamma_{|\ker A}$ is not trivial, then there exists $x \in \ker A$ such that $\gamma(x) \prec 0$ and $\alpha_i(x) = 0$, $\forall i$. Therefore, we may well assume that $\gamma_{|\ker A}$ is trivial, and restrict everything, including the map A, to W', but this is just the case for finite dimensional space.

Closely related to Farkas's Lemma is Gordan's Theorem. In the following formulation, we have to assume that V is nontrivial, i.e. $V \neq \{0\}$.

Gordan's Theorem 3.2 *Let $A : W \to F^m$ be a linear map and let $o : W \to V$ be the zero map, where $V \neq \{0\}$. Then either (A) there exists an $x \in W$ such that $\alpha_i(x) > 0, 1 \leq i \leq m$, or (B) there exist nonnegative vectors $u_1, \cdots, u_m \in V$, not all zero, such that $u_1\alpha_1 + \cdots + u_m\alpha_m = o$. The alternatives (A) and (B) exclude each other.*

Proof. Clearly (A) and (B) cannot be both true.

Assume that (B) is false, we will prove by induction that there exists x such that $\alpha_i(x) > 0$ for $i \leq m$.

For $m = 1$, that there does not exist $u_1 \succ 0$ such that $u_1\alpha_1 = o$ implies that α_1 is not the zero map, therefore there exists $x \in W$ such that $\alpha_1(x) > 0$.

Assume the result is true for the case $m - 1$, we will show the case m. By assumption, there exists $x_1 \in W$ such that $\alpha_i(x_1) > 0$ for all $i < m$. Now if $\alpha_m(x_1) > 0$, we would be finished. It remains to handle the cases $\alpha_m(x_1) = 0$ and $\alpha_m(x_1) < 0$.

Case 1: $\alpha_m(x_1) = 0$. By the same reasoning as in the base case, α_m cannot be the zero map, hence there exists $y \in W$ such that $\alpha_m(y) > 0$. Now if

$\alpha_i(y) \geq 0, \forall i < m$, then $x := x_1 + y$ satisfies $\alpha_i(x) > 0$ for all $i \leq m$. Otherwise, letting $a = \min_{i<m,\alpha_i(y)<0}\{-\alpha_i(y)^{-1}\alpha_i(x_1)\}$, and $\mu \in F$ be any number such that $0 < \mu < a$, it is straightforward to check that $x := x_1 + y\mu$ satisfies $\alpha_i(x) > 0$ for all $i \leq m$.

Case 2: $\alpha_m(x_1) < 0$. Define $\bar{\alpha}_i$ by $\bar{\alpha}_i = \alpha_i - \alpha_i(x_1)\alpha_m(x_1)^{-1}\alpha_m$ for $i < m$. Then $\bar{\alpha}_i$'s for $i < m$ clearly satisfy the induction hypothesis, therefore there exists $\bar{x}$ such that $\bar{\alpha}_i(\bar{x}) > 0$ for $i < m$. Letting now $x = \bar{x} - x_1\alpha_m(x_1)^{-1}\alpha_m(\bar{x})$, one checks easily that $\alpha_i(x) = \bar{\alpha}_i(\bar{x}) > 0$ for $i < m$ and $\alpha_m(x) = 0$, hence we have reduced to Case 1.

□

Remarks.

1. If $V = F = \mathbb{R}$, one gets the classical statement of Gordan's Theorem ([6]).

2. In the formulation of the generalized Gordan's Theorem, one does need to require that $V \neq \{0\}$. For example, let $V = \{0\}$, and $\alpha_2 = -\alpha_1$ for $m = 2$ in the theorem. Then neither condition (A) nor condition (B) can be satisfied.

Farkas implies Gordan.

Proof. Let $\alpha_1, \cdots, \alpha_m$ be linear maps from W to F which do not satisfy (B) of Gordan's Theorem. For each i, fix a $u_i' \succ 0$ and define $\gamma_i = -u_i'\alpha_i$. Then by assumption, there do no exist $u_j \succeq 0$, $j \neq i$ such that

$$\sum_{j \neq i} u_j \alpha_j = \gamma_i.$$

Therefore by Farkas's Lemma, there exists $x_i \in W$ such that $\alpha_j(x_i) \geq 0$ for $j \neq i$ and $\gamma_i(x_i) \prec 0$. The latter condition implies $\alpha_i(x_i) > 0$.

Now we define $x = x_1 + \cdots + x_m$. It is immediate that $\alpha_i(x) > 0$ for all $i \leq m$.

□

Conclusion.

We have proved Farkas's Lemma and Gordan's Theorem in the generalized form proposed by D. Bartl, using very down-to-earth argument, which closely follows Gale. We also proved that Farkas's Lemma implies Gordan's Theorem by a very simple argument. As mentioned in the introduction, in the classical setting, various key results are equivalent: for example, one can show that Gordan's Theorem implies Farkas's Lemma ([8]). It would be interesting to establish analogous result for the current setting, and in the adverse case, if the result is not true, find a suitable generalization for all the related theorems mentioned in the introduction.

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