International Mathematical Forum, Vol. 7, 2012, no. 25, 1241 - 1247

Some Results on Pseudo Generalized Quasi-Einstein Spaces

Eliza L. Bello

Institute of Mathematics
College of Science, University of the Philippines
Diliman, Quezon City 1101 Philippines

Richard S. Lemence

Ochanomizu University
2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610 Japan

Institute of Mathematics
College of Science, University of the Philippines
Diliman, Quezon City 1101 Philippines
rslemence@gmail.com

Dennis T. Leyson

Institute of Mathematics
College of Science, University of the Philippines
Diliman, Quezon City 1101 Philippines

Abstract

Generalizations of Einstein spaces are needed to have a deeper understanding of the global characteristics of the universe including its topology. One of the generalizations of this space was introduced by Shaikh and Jana in [6]. In the paper, they defined pseudo generalized quasi-Einstein spaces. Hence, in this paper, we give some results concerning pseudo generalized quasi-Einstein spaces.

Mathematics SUbject Classification: 53C25, 53C50

Keywords: Pseudo generalized quasi-Einstein spaces, W_2-curvature tensor, Codazzi type tensor

1 Introduction

To its present state, the evolution of the universe, can be divided into three phases [3]:

1. **Initial Phase.** This was just after the big bang when the effects of both viscosity and heat flux were quite pronounced.

2. **Intermediate Phase.** This was the phase when the effect of viscosity was no longer significant but the heat flux was still not negligible.

3. **Final Phase.** This phase extends to the present state of the Universe when both the effects of viscosity and the heat flux have become negligible and the matter content of the Universe may be assumed to be a perfect fluid.

To have a deeper understanding of the global characteristics of the universe including its topology, more generalized Einstein space are needed. For example, the generalized quasi-Einstein spacetime manifold, one of the existing generalizations of Einstein spaces, represents the intermediate phase of the evolution of the universe stated above while the quasi-Einstein spacetime manifold, another generalization of Einstein manifolds, corresponds to the final phase in the evolution.

Some have made generalizations of Einstein spaces including pseudo generalized quasi Einstein spaces ([6]) which are the main focus in this paper.

1.1 Related Work

Shaikh and Jana in [6] introduced another generalization of Einstein spaces. They defined a type of non-flat Riemannian manifold called *pseudo generalized quasi-Einstein manifold*, denoted by $P(GQE)_n$ where $n > 2$, and studied some properties of such manifold. They proved the existence theorem of $P(GQE)_n$ using some well-known results in [4] and [7]. They also proved some results concerning $P(GQE)_n$ admitting the conformal curvature tensor. Furthermore, they gave some interesting geometric properties of $P(GQE)_n$. Finally, they showed that a viscous fluid spacetime admitting heat flux and satisfying the Einstein's equation with a cosmological constant is a connected semi-Riemannian $P(GQE)_4$ and proved some algebraic properties of such a spacetime.

1.2 Our Results and Organization of the Paper

We study pseudo generalized quasi-Einstein spaces admitting W_2-curvature tensor.

The paper is organized as follows. In Section 2, we give the basic definitions and some properties needed in the preceding sections. In Section 3, we give our two main results: (1) on W_2-flat pseudo generalized quasi-Einstein spaces; and, (2) on pseudo generalized quasi-Einstein spaces with constants associated scalars, with Codazzi type structure tensor; and, with generators with associated 1-forms being recurrences.

2 Basic Definitions

A Riemannian manifold $M = (M^n, g)$, $n > 2$ is called an *Einstein space* if the Ricci tensor $S(X,Y) = \alpha g(X,Y)$, for smooth vectors X and Y on M.

In [1], Chaki and Maity defined a quasi-Einstein manifold. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be *quasi-Einstein manifold*, if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1}$$

where α and β are scalars of which $\beta \neq 0$ and A is a nowhere vanishing 1-form such that $g(X,\rho) = A(X),\ g(\rho,\rho) = A(\rho) = 1$ for the associated vector field ρ. The 1-form A is called the associated 1-form and the unit vector field ρ is called the generator of the manifold. An n-dimensional manifold of this kind is denoted by $(QE)_n$. The scalars α and β are known as the associated scalars.

As a generalization of quasi-Einstein manifold, Chaki [2] introduced the notion of generalized quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be *generalized quasi-Einstein manifold* if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)], \tag{2}$$

where α, β, γ are scalars of which $\beta \neq 0$, $\gamma \neq 0$, A, B are non-zero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all X and ρ, μ are two unit vector fields mutually orthogonal to each other. In such a case α, β and γ are called the associated scalars, A, B are called the associated 1-forms and ρ, μ are the generators of the manifold. Such an n-dimensional manifold is denoted by $G(QE)_n$.

A Riemannian manifold (M^n, g) $(n > 2)$ is said to be a pseudo generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta D(X,Y) \tag{3}$$

where α, β, γ and δ are nonzero scalars, A and B are two nonzero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X and ρ, μ are mutually orthogonal unit vectors and D is a symmetric (0,2) tensor with zero trace, which satisfies the condition $D(X, \rho) = 0$ for all vector fields X.

In 1970, Pokhariyal and Mishra [5] introduced several tensor fields in a Riemannian manifold and studied their properties. Their list includes W_2-curvature tensor on a manifold (M^n, g) which is defined by

$$W_2(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)S(Y,T) - g(Y,Z)S(X,T)\right] \ , \tag{4}$$

where R is the Riemannian curvature tensor.

3 Main Results

Theorem 1. *In A W_2-flat pseudo generalized quasi-Einstein manifold, the relation*

$$A(Y)D(X,\mu) = A(X)D(Y,\mu)$$

holds if and only if either the vector fields ρ and μ corresponding to the 1-forms A and B respectively are codirectional, the associated scalars γ and β are equal, or $\epsilon = 0$.

Proof. Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal frame field at any point of the manifold. Then setting $X = Y = e_i$ in (3) and taking the summation over i, $1 \leq i \leq n$, we obtain

$$r = n\alpha + \beta + \gamma \tag{5}$$

where r is the scalar curvature of the manifold.

If we consider a pseudo generalized quasi-Einstein manifold which is W_2-flat, *i.e.*, $W_2 = 0$, then from (3) and (4), we obtain

$$\begin{aligned} R(X,Y,Z,U) &= \frac{1}{n-1}\{g(Y,Z)\left[\alpha g(X,U) + \beta A(X)A(U) + \gamma B(X)B(U)\right. \\ &\quad \left. + \,\delta D(X,U)\right] - g(X,Z)\left[\alpha g(Y,U) + \beta A(Y)A(U)\right. \\ &\quad \left. + \,\gamma B(Y)B(U) + \delta D(Y,U)\right]\} \\ &= \frac{1}{n-1}\{\alpha\left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right] \\ &\quad +\beta A(U)\left[g(Y,Z)A(X) - g(X,Z)A(Y)\right] \\ &\quad +\gamma B(U)\left[g(Y,Z)B(X) - g(X,Z)B(Y)\right] \\ &\quad + \delta\left[g(Y,Z)D(X,U) - g(X,Z)D(Y,U)\right]\} \ . \end{aligned}$$

Setting $Z=\rho$ and $U=\mu$, we get

$$\begin{aligned}
R(X,Y,\rho,\mu) &= \frac{1}{n-1}\{\alpha\,[g(Y,\rho)g(X,\mu)-g(X,\rho)g(Y,\mu)] \\
&\quad +\beta A(\mu)\,[g(Y,\rho)A(X)-g(X,\rho)A(Y)] \\
&\quad +\gamma B(\mu)\,[g(Y,\rho)B(X)-g(X,\rho)B(Y)] \\
&\quad +\delta\,[g(Y,\rho)D(X,\mu)-g(X,\rho)D(Y,\mu)]\}\ .
\end{aligned}$$

Since $A(\mu)=g(\rho,\mu)=B(\rho)=0$, $A(\rho)=g(\rho,\rho)=\epsilon$, $B(\mu)=g(\mu,\mu)=\epsilon$, we have

$$\begin{aligned}
R(X,Y,\rho,\mu) &= \frac{1}{n-1}\{\alpha\,[A(Y)B(X)-A(X)B(Y)] \\
&\quad +\epsilon\gamma\,[A(Y)B(X)-A(X)B(Y)] \\
&\quad +\delta\,[A(Y)D(X,\mu)-A(X)D(Y,\mu)]\} \\
&= \frac{1}{n-1}\{[\alpha+\epsilon\gamma]\,[A(Y)B(X)-A(X)B(Y)] \\
&\quad +\delta\,[A(Y)D(X,\mu)-A(X)D(Y,\mu)]\}\ .
\end{aligned}$$

Similarly, if we set $Z=\mu$ and $U=\rho$, we get

$$\begin{aligned}
R(X,Y,\mu,\rho) &= \frac{1}{n-1}\{\alpha\,[B(Y)A(X)-B(X)A(Y)] \\
&\quad +\beta\epsilon\,[B(Y)A(X)-B(X)A(Y)] \\
&\quad +\delta\,[B(Y)D(X,\rho)-B(X)D(Y,\rho)]\}\ .
\end{aligned}$$

Using the property that $R(X,Y,\rho,\mu)=-R(X,Y,\mu,\rho)$, we have

$$\begin{aligned}
&[\alpha+\epsilon\gamma]\,[A(Y)B(X)-A(X)B(Y)]+\delta\,[A(Y)D(X,\mu)-A(X)D(Y,\mu)] \\
=\ &[\alpha+\beta\epsilon]\,[A(Y)B(X)-A(X)B(Y)]+\delta\,[B(X)D(Y,\rho)-B(Y)D(X,\rho)]\ .
\end{aligned}$$

Because $D(X,\rho)=0$ for all vector fields X, we obtain

$$[\gamma\epsilon-\beta\epsilon]\,[A(Y)B(X)-A(X)B(Y)]+\delta\,[A(Y)D(X,\mu)-A(X)D(Y,\mu)]=0\ . \tag{6}$$

If the vector fields μ and ρ are codirectional then

$$A(X)B(Y)=A(Y)B(X)\ . \tag{7}$$

Applying (7) in (6), we get

$$A(Y)D(X,\mu)=A(X)D(Y,\mu) \tag{8}$$

since $\delta\neq 0$.

Conversely, if the relation (8) holds then either (7) holds, $\epsilon=0$ or $\gamma=\beta$. $\square$

Theorem 2. *If in a pseudo generalized quasi-Einstein manifold the associated scalars are constants, the structure tensor is of Codazzi type and the generators ρ and μ are vector fields with the associated 1-forms A and B not being the 1-forms of recurrences then the manifold is W_2-conservative.*

Proof. From (4), we have

$$(\text{div } W_2)(Y,Z)U = (\text{div } R)(Y,Z)U + \frac{1}{2(n-1)}\left[dr(Z)g(Y,U) - dr(Y)g(Z,U)\right] \tag{9}$$

where 'div' denotes the divergence.

It is known that in a Riemannian manifold,

$$(\text{div } R)(Y,Z)U = (\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U).$$

Using this relation, (9) now takes the form

$$\begin{aligned}(\text{div } W_2)(Y,Z)U &= (\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U) \\ &\quad + \frac{1}{2(n-1)}\left[dr(Z)g(Y,U) - dr(Y)g(Z,U)\right] .\end{aligned} \tag{10}$$

If the associated scalars α, β, γ, and δ are constants, then (5) means the scalar curvature r is constant and therefore $dr(X) = 0$ for all X. Hence, (10) yields

$$(\text{div } W_2)(Y,Z)U = (\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U) . \tag{11}$$

From (3) and because α, β, γ, δ are constants, it follows that

$$\begin{aligned}(\nabla_Y S)(Z,U) &= \beta\left[A(Z)(\nabla_Y A)U + A(U)(\nabla_Y A)Z\right] \\ &\quad + \gamma\left[B(Z)(\nabla_Y B)U + B(U)(\nabla_Y B)Z\right] \\ &\quad + \delta(\nabla_Y D)(Z,U) .\end{aligned} \tag{12}$$

Assume that the structure tensor D of such manifold is of Codazzi type. Thus, for all vector fields Y, Z, U, we have

$$(\nabla_Y D)(Z,U) = (\nabla_Z D)(Y,U) . \tag{13}$$

Substituting (12) and (13) on (11) yields

$$\begin{aligned}(\text{div } W_2)(Y,Z)U &= \beta\{[A(Z)(\nabla_Y A)U + A(U)(\nabla_Y A)Z] \\ &\quad - [A(Y)(\nabla_Z A)U + A(U)(\nabla_Z A)Y]\} \\ &\quad + \gamma\{[B(Z)(\nabla_Y B)U + B(U)(\nabla_Y B)(Z)] \\ &\quad - [B(Y)(\nabla_Z B)U + B(U)(\nabla_Z B)Y]\} .\end{aligned} \tag{14}$$

Now, if ρ and μ of the manifold are recurrent vector fields then we have $\nabla_Y \rho = \pi_1(Y)\rho$ and $\nabla_Y \mu = \pi_2(Y)\mu$, where π_1 and π_2 are 1-forms different from A and B. Then

$$(\nabla_Y A)Z = g(\nabla_Y \rho, Z) = g(\pi_1(Y)\rho, Z) = \pi_1(Y)A(Z) \tag{15}$$
$$(\nabla_Y B)Z = g(\nabla_Y \mu, Z) = g(\pi_2(Y)\rho, Z) = \pi_2(Y)B(Z) \tag{16}$$

By virtue of (15) and (16), (14) becomes

$$\begin{aligned}(\text{div } W_2)(Y,Z)U &= 2\beta\left[A(Z)A(U)\pi_1(Y) - A(Y)A(U)\pi_1(Z)\right] \\ &\quad +2\gamma\left[B(Z)B(U)\pi_2(Y) - B(Y)B(U)\pi_2(Z)\right] .\end{aligned} \tag{17}$$

Because $g(\rho,\rho) = g(\mu,\mu) = 1$, it follows that $(\nabla_Y A)(\rho) = g(\nabla_Y \rho, \rho) = 0$ and thus (15) reduces to $\pi_1(Y) = 0$ for all Y. In a similar manner for (16), we have $\pi_2(Y) = 0$. Hence, from (17), we obtain $(\text{div } W_2)(Y,Z)U = 0$, *i.e.,* the manifold is W_2-conservative. □

Acknowledgment. R.S. Lemence is partially supported by the Special Coordination Funds for Promoting Science and Technology, Japan.

References

[1] Chaki, M. C. and Maity, R. K., *On quasi-Einstein manifolds*, Publ. Math. Debrecen, **57** (2000), 297–306.

[2] Chaki, M. C., *On generalized quasi-Einstein manifolds*, Publ. Math. Debrecen, **58** (2001), 638–691.

[3] Guha, S., *On quasi Einstein and generalized quasi-Einstein manifolds*, Facta Universitatis (Series: Mechanics, Automatic Control and Robotics), **3(14)** (2003), 821 –842.

[4] B. O'Neill. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, Inc., 1983.

[5] G.P. Pokhariyal. *Curvature Tensors and Their Relativistic Significance*. Yokohama Math. Journal, Vol. 21: 115-119, 1973.

[6] A.A, Shaikh and S.K. Jana. *On pseudo generalized quasi-Einstein manifods*, Tamkang J. of Mathematics, **39(1)** (2008), 9–24.

[7] M. Spivak. *A comprehensive introduction to Differential Geometry*. Publish and Perish, Vol. I, 1970.

Received: November, 2011