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Procesi Extensions of Filtered and Graded Rings Applied to the Micro-Affine Schemes

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Abstract

In this paper we continue our study of filtered and graded Procesi extensions of rings, that was introduced in [15]. After a brief study of general features concerning filtered and graded Procesi extensions at the level of graded Rees rings and associated graded rings, we turn to the behaviour of filtered and graded Procesi extensions towards to the micro-affine schemes. A number of results concerning these notations are given.

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1 Introduction

There are many different ways of describing the ring extensions and their applications, several of these were thought to be different in some papers, see [4, 6, 7, 8, 18].

For basic notations, conventions and generalities, which we need here in this paper, we refere to [1-5, 9, 10, 16, 17].

If $\varphi : R \to S$ is a ring homomorphism of not necessarily commutative rings, the prime ideal structures of R and S are, in general, rather poorly related, see [6, 7, 8, 11, 12, 13, 14]. It appears that if one restricts to so-called filtered and graded Procesi extension of rings, as we see in this paper, then things improve considerably over the filtered and graded levels. Also, the usefulness of the topological spec-map appears in the study of ring extensions, the theory of schemes which centers around sheaves and in algebraic geometry.

We continue our study of filtered and graded Procesi extensions of rings and show that theses extensions behave well, over the filtered and graded levels, in constructing geometric spaces. We investigate the effect of existence of the extension $\varphi : R \to S$; a filtered ring homomorphism, with $S = \varphi(R).S^R$ on the filtered ring S. On considering two filtrations $F'S, F''S$ on $S = \varphi(R).S^R$, we can study the transfer of more properties from the given filtrations FR, FS to $F'S, F''S$. This is as in the first section. Next, by using the Rees graded level and for all n, we prove that if S is a filtered Procesi extension of R then $\bar{\tilde{S}}(n) = \tilde{S}/X^n\tilde{S}$ is again a graded Procesi extension of $\tilde{R}/X^n\tilde{R} = \bar{\tilde{R}}(n)$, here Proposition 3.1. in [15] is a special case. In the final section of the paper, on considering the micro-structure sheaves we will show that our Procesi extension can be applied to the micro-affine schemes.

2 Filtered Procesi Extensions

Throughout this paper R and S will denote filtered rings with units and $\mathbb{Z}$-filtrations $FR = \{F_nR\}_{n\in\mathbb{Z}}$ on R, $FS = \{F_nS\}_{n\in\mathbb{Z}}$ on S. R-filt will denote the category of left filtered R-modules.

Let $\varphi : R \to S$ be a filtered ring homomorphism in R-filt, then φ is said to be filtered Procesi extension if $S = \varphi(R).S^R$ where $S^R = \{s \in S : \varphi(r)s = s\varphi(r) \forall r \in R\}$. For $S = \varphi(R).S^R$, there are two filtrations on it with respect to the given filtrations FR, FS, i.e., $F'S = \{F'_nS\}_{n\in\mathbb{Z}}$; with $F'_nS = F_n(\varphi(R)).S^R = (\varphi(R) \cap F_nS).S^R$ and $F''S = \{F''_nS\}_{n\in\mathbb{Z}}$; with $F''_nS = F_n(\varphi(R)).S^R = \varphi(F_nR).S^R$. It is equally straight forward to derive the following:

Lemma 2.1. *under the assumptions and notations mentioned above we have:*
(a) S, with the filtration $F'S$, is a filtered ring. Also, it is a filtered ring with respect to $F''S$,
(b) S, with the filtration $F'S$, is a left filtered R-module. Also, it is a left filtered R-module with respect to $F''S$,
(c) $HOM_{FR,FS}(R,S) \subseteq HOM_{FR,F'S}(R,S)$ and $HOM_{FR,FS}(R,S) \subseteq HOM_{FR,F''S}(R,S)$,
(d) For every $n \in \mathbb{Z}, F''_nS \subseteq F'_nS$, and
(e) $\varphi(1_R) = 1_S$.

A morphism $f : M \to N$, in R-filt, is said to be strict if $f(F_nM) = f(M) \cap F_nN = F_nf(M)$, for all $n \in \mathbb{Z}$. For a complete information of filtered *and graded ring theory, the reader is refered to [1, 5, 9, 10].*

The following results are some more properties of filtered Procesi extensions.

Proposition 2.2. *Let $\varphi : R \to S$ be a filtered Procesi extension as above.*
(a) If φ is strict, then $F'S = F''S$ on S.
(b) If φ is a monomorphism and R a strongly filtered ring, in the sense that

$$F_nR.F_mR = F_{n+m}R, \quad for\, all \quad n, m \in \mathbb{Z},$$

then

$$F'_nS = F''_nS \cong F_nR.S^R \quad for\, all \quad n \in \mathbb{Z},$$

and $F'S, F''S$ are strong filtrations on S.
(c) If φ is a monomorphism and I any two sided ideal in R, then $IS = SI$.
(d) If φ is strict and R a strongly filtered ring, then S is strongly filtered with respect to $F'S = F''S$.
(e) $\varphi(Z(R)) \subseteq Z(S) \subseteq S^R$, where $Z(R)$ is the commutative subring in R of all the central elements, with the induced filtration.
(f) If φ is an epimorphism and S a strongly filtered ring with respect to FS, then $S = S.Z(S)$ is strongly filtered with respect to

$$F''S = F'S; F'_nS = F''_nS = F_nS.Z(S), \quad for\, all \quad n \in \mathbb{Z}.$$

Proof. Straight forward.

A filtration FM on $M \in R$-filt is said to be discrete if there is an integer α such that $F_nM = 0$ for all $n < \alpha$, separated if $\bigcap F_nM = \{0\}$ and exhaustive if $M = \bigcup_{n\in\mathbb{Z}} F_nM$. Finally, M is said to be filtered complete if $M \cong M^{\wedge_F} = \varprojlim_n M/F_nM$.

In other words, M is complete if FM is separated and all Cauchy sequences in the FM-topology of M converge, see [1, 2, 5]. ☐

Proposition 2.3. *With notations as above.*
(a) If FS and FR are discrete then $F'S$ and $F''S$ are discrete.
(b) If FS and FR are separated then $F'S$ and $F''S$ are separated.
(c) If FS and FR are exhaustive then $F'S$ and $F''S$ are exhaustive.
(d) If FS and FR are exhaustive, separated and complete then S is a complete filtered at $F'S$ and at $F''S$.

Proof. (a) We may take $\alpha = \alpha_{FR} = \alpha_{FS} = \alpha_{F'S} = \alpha_{F''S}$. Then we have

$$F'_nS = (F_nS \bigcap \varphi(R)).S^R = (0 \bigcap \varphi(R)).S^R = 0,$$

and

$$F''_nS = \varphi(F_nR).S^R = \varphi(0).S^R = 0, \quad for\, all \;\; n < \alpha.$$

(b) If $t = \bigcap_{n\in\mathbb{Z}} F'_nS = \bigcap_{n\in\mathbb{Z}} (\varphi(R) \bigcap F_nS).S^R$ and $\bigcap F_nS = \{0\}$, then $t = 0$, i.e., $\bigcap_{n\in\mathbb{Z}} F'_nS = \{0\}$. In similar, if $t \in \bigcap_{n\in\mathbb{Z}} F''_nS = \bigcap_{n\in\mathbb{Z}} (\varphi(F_nR).S^R)$ and $\bigcap F_nR = \{0\}$, then t=0 and therefore $\bigcap_{n\in\mathbb{Z}} F''_nS = \{0\}$.
(c) Since $\bigcup_{n\in\mathbb{Z}} F_nS = S$ and $\bigcup_{n\in\mathbb{Z}} F_nR = R$, then

$$\bigcup_{n\in\mathbb{Z}} F'_nS = \bigcup_{n\in\mathbb{Z}}((F_nS \bigcap \varphi(R)).S^R) = (\bigcup_{n\in\mathbb{Z}} F_nS \bigcap \varphi(R)).S^R = \varphi(R).S^R = S.$$

Also

$$\bigcup_{n\in\mathbb{Z}} F''_nS = \bigcup_{n\in\mathbb{Z}}(\varphi(F_nR).S^R) = \varphi(R).S^R = S.$$

This yields the assertion.
(d) By using (b) and (c), we conclude that all Cauchy sequences in the $F'S-$ $(F''S-)$ topology of S converge.

Now, let $0 \neq t = s_{-1}s^r \in F'_{-1}S = (\varphi(R) \bigcap F_{-1}S).S^R; s_{-1} \in \varphi(R) \bigcap F_{-1}S$ and $s^r \in S^R$. Then

$$1 \in F_{-(n+1)}S + (1-t)((\varphi(R) \bigcap F_0S).S^R).$$

By using the closed condition, in the sense of Björk for FS, see [3], and the Jacobson condition $F_{-1}S \subset J(F_0S)$ at FS, we have that $1-t$ is an invertible in $F'_0S = (\varphi(R) \bigcap F_0S).S^R$. Hence $t \in J(F'_0S)$ and $F'_{-1}S \subset J(F'_0S)$. Therefore we have proved the following: $\square$

Proposition 2.4. *With notations as above.*
If FS, FR satisfy the closed condition of Björk and the Jacobson condition, then $F'S, F''S$ satisfy the Jacobson condition.

3 Graded Procesi Extensions

If $\varphi : R \to S$ is a filltered Procesi extension with $S = \varphi(R).S^R; F'S = \{(\varphi(R) \bigcap F_nS).S^R\}_{n\in\mathbb{Z}}$ or $F''S = \{\varphi(F_nR).S^R\}_{n\in\mathbb{Z}}$ as above then, in view of Proposition 3.1. in [15], we have two associated graded Procesi extensions: $\tilde{\varphi} : \tilde{R} \to \tilde{S}$ of Rees rings $\tilde{R} = \bigoplus F_nR \cong \Sigma F_nR.X^n, \tilde{S} = \bigoplus F_nS \cong \Sigma F_nS.X^n$ with $\tilde{S} = \tilde{\varphi}(\tilde{R}).\tilde{S}^{\tilde{R}} = (\varphi(R).S^R)^{\sim}$ and $T = \bar{\tilde{\varphi}} = G(\varphi) : G(R) \longrightarrow G(S)$ of associated graded rings $G(R) = \bigoplus F_nR/F_{n-1}R \cong \tilde{R}/X\tilde{R} = \bar{\tilde{R}}, G(S) = \bigoplus F_nS/F_{n-1}S \cong \tilde{S}/X\tilde{S} = \bar{\tilde{S}}$ with $\bar{\tilde{S}} = T(\bar{\tilde{R}}).\bar{\tilde{S}}^{\bar{\tilde{R}}}$. Hence,

$$X_{\tilde{S}^{F''}} = X_{\tilde{S}^{F}}.1 \in (\tilde{S}^{F''})_1 = \varphi(F_1R).S^R \, \textit{too.}$$

This and (a) of Proposition 2.2 ential:

Proposition 3.1. *With assumptions and notations as before.*

Let $\varphi : R \longrightarrow S$ be a strict filtered Procesi extension with $S = \varphi(R).S^R$; $F'S$ or $F''S$ on S. Then $\bar{\tilde{R}} \longrightarrow \bar{\tilde{S}}; \bar{\tilde{\varphi}}(\bar{\tilde{r}}) = \tilde{\varphi}(\tilde{r}) + X\tilde{S}$ for all $\bar{\tilde{r}} \in \bar{\tilde{R}}$, is a graded Procesi extension in the category $\bar{\tilde{R}} - gr$ with $\bar{\tilde{S}} = \bar{\tilde{\varphi}}(\bar{\tilde{R}}).\bar{\tilde{S}}^{\bar{\tilde{R}}}$ and $G_{F'}(S) = G_{F''}(S)$.

Remark 3.2. *Our main aim in this paper is to show that these extensions per- mit us to construct the Procesi extensions of micro-affine schemes. So, for con- venience's sake in what follows, we only fix $F'S$ on S with respect to the given filtrations FR, FS.*

Consider the following commutative diagram; for $n > 1$:

$$\begin{tikzcd}
\tilde{R} \arrow[rrr, "\tilde{\varphi}"] \arrow[dr] \arrow[dd] & & & \tilde{S} \arrow[dl] \arrow[dd] \\
& \tilde{R}/X\tilde{R} \arrow[r] & \tilde{S}/X\tilde{S} & \\
\bar{\tilde{R}}(n) = \tilde{R}/X^n\tilde{R} \arrow[ur] \arrow[rrr, "T_{(n)}"] & & & \tilde{S}/X^n\tilde{S} = \bar{\tilde{S}}(n) \arrow[ul]
\end{tikzcd}$$

$T_{(n)}$ may be defined as: $T_{(n)}(\bar{\tilde{r}}(n)) = \tilde{\varphi}(\tilde{r}) + X^n\tilde{S}$ for all $\bar{\tilde{r}}(n) \in \bar{\tilde{R}}(n)$

Now, let $\bar{\tilde{t}}(n) = \tilde{s} + X^n\tilde{S} \in \bar{\tilde{S}}(n); \tilde{s} \in \tilde{S} = \tilde{\varphi}(\tilde{R}).\tilde{S}^{\tilde{R}}$ and $\tilde{s} = \tilde{\varphi}(\tilde{r})\tilde{z} = \tilde{z}\tilde{\varphi}(\tilde{r}); \tilde{z} \in \tilde{S}^{\tilde{R}}$. Then

$$\tilde{z}\tilde{\varphi}(\tilde{r}) - \tilde{\varphi}(\tilde{r})\tilde{z} \in X^n\tilde{S}$$

Therefore

$$\bar{\tilde{t}}(n) = \tilde{z}\tilde{\varphi}(\tilde{r}) + X^n\tilde{S} \in T_{(n)}(\bar{\tilde{R}}(n))(\bar{\tilde{S}}(n))^{\bar{\tilde{R}}(n)}$$

Conversely, let $\bar{\tilde{s}}(n) \in T_{(n)}(\bar{\tilde{R}}(n))(\bar{\tilde{S}}(n))^{\bar{\tilde{R}}(n)}$. *Then*

$$\begin{aligned}\bar{\tilde{s}}(n) &= (\tilde{\varphi}(\tilde{r}) + X^n\tilde{S}).\bar{\tilde{z}}(n) = (\tilde{\varphi}(\tilde{r}) + X^n\tilde{S})(\tilde{z} + X^n\tilde{S}) \\ &= \tilde{\varphi}(\tilde{r})\tilde{z} + X^n\tilde{S} = \tilde{z}\tilde{\varphi}(\tilde{r}) + X^n\tilde{S} \in \bar{\tilde{S}}(n).\end{aligned}$$

Hence, we conclude that $\bar{\tilde{S}}(n) = T_{(n)}(\bar{\tilde{R}}(n))(\bar{\tilde{S}}(n))^{\bar{\tilde{R}}}$ *and* $T_{(n)}$ *is a graded Procesi extension in* $(\tilde{R} - gr)\bar{\tilde{R}}(n) - gr$.
Therefore we have proved the following result:

Proposition 3.3. *Let* $\varphi : R \longrightarrow S$ *be a filtered Procesi extension in R-filt such that*

$$S = \varphi(R).S^R; F'S = \{(F_nS \bigcap \varphi(R)).S^R\}_{n\in\mathbb{Z}}.$$

Then φ*, for all* $n > 1$*, induces a graded Procesi extension* $T_{(n)} : \bar{\tilde{R}}(n) \longrightarrow \bar{\tilde{S}}(n)$ *such that* $\bar{\tilde{S}}(n) = T(n)(\bar{\tilde{R}}(n))(\bar{\tilde{S}}(n))^{\bar{\tilde{R}}(n)}$.

Remark 3.4. *One may think about the transfer of regularity in graded Procesi extension rings. We hope to investigate this problem in forthcoming work. Now, we can mention the general result of this section as follows:*

Proposition 3.5. *Let* $\varphi : R \longrightarrow S$ *be a filtered Procesi extension with*

$$S = \varphi(R).S^R; F'S = \{(\varphi(R) \bigcap F_nS).S^R\}_{n\in\mathbb{Z}}$$

and $I \trianglelefteq R, J \trianglelefteq S$ *two sided filtered ideals with the induced filtrations in* R, S *respectively such that* $\varphi(I) \subseteq J$*. Then* $\varphi_I : \frac{R}{I} \longrightarrow \frac{S}{J}$*, defined by* $\varphi_I(r + I) = \varphi(r) + J$ *for all* $\bar{r} \in \frac{R}{I}$ *is a filtered Procesi extension with*

$$\frac{S}{J} = \varphi_I\left(\frac{R}{I}\right).\left(\frac{S}{J}\right)^{R/I} = \frac{\varphi(R)}{J}.\left(\frac{S}{J}\right)^{R/I}$$

4 Filtered and Graded Procesi Extension of Micro-affine Schemes

Throughout of this section, R and S will be filtered rings with units such that $G(R), G(S)$ are Noetherian domains.

Let us endow the graded prime spectrum $X = Spec^g(G(R))$ (similar to $Y = Spec^g(G(S)))$ with the so-called Zariski topology, by letting the open sets (then basic affine Noetherian open sets, see [12, 13, 14]) for this topology to be the sets $X(f) = \{p \in X : f \notin p\}$, where f runs through the homogeneous elements of $G(R)$.

In general, $Spec^g G(R)$ is not a scheme. However in case $G(R)$ is positively graded, then we write $Proj(G(R))$ for the Zariski open subspace of X consisting of the graded prime ideals not containing $G(R)_+ = \bigoplus_{n>0} G(R)_n$, and in this case the closed set $V(G(R)_+)$ in X is nothing but $Spec(G(R)_0)$. Therefore $p(X) = Proj(G(R)) = \{p \in X : G(R)_n \nsubseteq p \text{ for some } n > 0\}$. It is clear that $X = p(X)$ if and only if $G(R)_n G(R)_{-n} = G(R)$ for all $n > 0$. Also, a ring morphism $\varphi : R \longrightarrow S$ does not even induces a map $Spec(S) \longrightarrow Spec(R)$, since $\varphi^{-1}(q)$ is not necessarily prime in R if q is a prime ideal of S. However, this works if φ is a Procesi extension. A first positive result will be given as follows:

Proposition 4.1. *([15-4.1]).*

The filtered Procesi extension $\varphi : R \longrightarrow S$, such that $S = \varphi(R).S^R$ induces a continuous morphism

$$^aT = {}^a(G(\varphi)) : Spec^g(G(S)) = Y \longrightarrow X = Spec^g(G(R)), p \longmapsto T^{-1}(p) = G(\varphi)^{-1}(p).$$

Let us investigate and construct the behaviour of the Procesi extension with respect to affine schemes on the graded level. This enables us to study the filtered micro-level.

If $f \in h(G(R) \bigcap Z(G(R))), p \in Y$ then $T(f) \in h(G(S) \bigcap Z(G(S))), q = {}^aT(p) = T^{-1}(p) \in X$.

Denote by $\underline{O}^g_X, \underline{O}^g_Y$ the graded localization structure sheaves over X, Y respectively.

Hence, the graded Procesi extension $T = G(\varphi)$ induces two graded Procesi extensions of graded rings:

$$T_f : G(R)_f = \underline{O}^g_X(X(f)) \longrightarrow \underline{O}^g_Y(Y(T(f))) = G(S))_{T(f)}$$

of graded rings of fractions at $f, T(f)$ respectively, $T_p : \underline{O}^g_{X,q} \longrightarrow \underline{O}^g_{Y,p}$ of stalks of $\underline{O}^g_X, \underline{O}^g_Y$ at q, p respectively. We then have the following result:

Proposition 4.2. *(cf. [15] Proposition 4.2.).*

The filtered Procesi extension of filtered rings $\varphi : R \longrightarrow S$, such that $S = \varphi(R).S^R$ induces a graded Procesi extension $({}^aT, T_{sheaf}) : (Y = Spec^g G(S), \underline{O}^g_Y \longrightarrow (X = Spec^g G(R), \underline{O}^g_X)$ of graded affine schemes.

Now, if $\tilde{f}$ is the multiplicative (Ore) set in $\tilde{R}$ associated to f in $G(R)$ then we let, for all $n, \tilde{f}(n)$ its image in $\bar{\tilde{R}} = \tilde{R}/X^n\tilde{R}$. Denote by $\underline{\tilde{O}}^{(n)}_X, \underline{\tilde{O}}^{(n)}_Y$ the *graded localization structure sheaves over* X, Y *respectively at the level* n. *Hence, from Proposition 3.3 and Proposition 4.2, we have two graded Procesi extensions rings:* $\underline{T}_{(n)\bar{\tilde{f}}(n)} : \bar{\tilde{R}}_{\bar{\tilde{f}}(n)} \longrightarrow \bar{\tilde{S}}_{T_{(n)}(\bar{\tilde{f}}(n))}$ *of graded rings of fractions of* $\bar{\tilde{R}}, \bar{\tilde{S}}$ *at* $\bar{\tilde{f}}(n), T_{(n)}(\bar{\tilde{f}}(n))$ *respectively. These extensions are compatible with the restriction graded homomorphisms of the graded structure sheaves* $\underline{\tilde{O}}_X^{(n)}, \underline{\tilde{O}}_Y^{(n)}$. *And a graded Procesi extension of their stalks* $T_{(n)p} : \underline{\tilde{O}}_{X,q}^{(n)} \longrightarrow \underline{\tilde{O}}_{Y,p}^{(n)}$. *Here, we have to point out that the graded localization functors, associated to* $p \in Y$ *and* ${}^aT(p) = q \in X$ *are exact. For one moment, we can write the following:*

Proposition 4.3. *With notations and conventions as above.*

$$(Y = Spec^g(G(S)), \underline{\tilde{O}}_Y^{(n)}) \longrightarrow (X = Spec^g(G(R)), \underline{\tilde{O}}_X^{(n)})$$

, for all n, *is a graded Procesi extension of graded affine schemes.* *Take the inverse limit at the graded sense with respect to the above result. Denote by* $\underline{\tilde{O}}_X^{\mu}, \underline{\tilde{O}}_Y^{\mu}$ *the graded microlocalization structure sheaves over* X, Y *respectively of graded rings of sections*

$$\underline{\tilde{Q}}_{\tilde{f}}^{\mu}(\tilde{R}) = \lim_{\underset{n}{\longleftarrow}} {}^g\bar{\tilde{R}}_{\bar{\tilde{f}}(n)}, \underline{\tilde{Q}}_{\tilde{\varphi}(\tilde{f})}^{\mu}(\tilde{S}) = \lim_{\underset{n}{\longleftarrow}} {}^g\bar{\tilde{S}}_{T_{(n)}(\bar{\tilde{f}}(n))}$$

and having as completed stalks at p, ${}^aT(p)$ *the graded local rings* $\tilde{Q}_p^{\mu}(\tilde{R}), \tilde{Q}_q^{\mu}(\tilde{S})$, *see [11]. Also, then we have the following:*

Proposition 4.4.

$$(Y = Spec^g(G(S)), \underline{\tilde{O}}_Y^{\mu}) \longrightarrow (X = Spec^g(G(R)), \underline{\tilde{O}}_X^{\mu})$$

is a graded Procesi extension of micro-affine schemes. This is the micro-Rees level. Let us finish this section with a result concerning the filtered Procesi extension of micro-affine schemes.

Put $Q_f^{\mu}(R) = \tilde{Q}_{\tilde{f}}^{\mu}(\tilde{R})/(1 - X) = \tilde{Q}_{\tilde{f}}^{\mu}(\tilde{R})$; *i.e. the microlocalization at the level of the filtered ring is obtained by dehomogenization of the construction at Rees ring level, see [1]. Denote by* $\underline{O}_X^{\mu}, \underline{O}_Y^{\mu}$ *the filtered micro-structure sheaves defined by associating to* $X(f)$, *for a homogenous* f, *the rings of sections* $Q_f^{\mu}(R), Q_{T(f)}^{\mu}(S)$. *The stalks of* $\underline{O}_X^{\mu}, \underline{O}_Y^{\mu}$ *at* $p, q = {}^aT(p)$ *are filtered rings* $\underline{O}_{X,p}^{\mu}, \underline{O}_{Y,q}^{\mu}$ *such that*

$$(\underline{O}_{X,p}^{\mu})^{\wedge} = Q_p^{\mu}(R), (\underline{O}_{Y,q}^{\mu})^{\wedge} = Q_q^{\mu}(S).$$

these being the microlocalizations of R, S *with respect to the homogeneous Ore sets* $h(G(R) - p), h(G(S) - q)$ *respectively. In this filtered setting, the condition of Zariski filtered ring, may be necesarily.*

Proposition 4.5. *With assumptions and notations as before.*

The filtered Procesi extension $\varphi : R \to S$ with $S = \varphi(R).S^R$ induces a filtered Procesi extension

$$(Y = Spec^g(G(S)), \underline{O}^{\mu}_{Y}) \to (X = Spec^g(G(R)), \underline{O}^{\mu}_{X})$$

of the filtered micro-affine schemes.

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