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Riemann-Liouville Sesquiderivative

and Sesquiintegral

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Abstract

In this paper we aim to derived sesquiderivative and sesquiintegral of some interesting functions like algebraic, trigonometric, logarithmic and exponential. Sesquiderivative and Sesquiintegral denote the operation performed by the, $(\frac{d^{3/2}}{dx^{3/2}})$ and $(\frac{d^{-3/2}}{dx^{-3/2}})$ operators. To obtain sesquiderivative , apply Riemann-Liouville integral formula with q = 3/2 then differentiate the obtained result three whole time .

Keywords: Fractional Calculus, Fractional Integral operator, Fractional differential operator, Sesquiderivative and Sesquiintegral.

1 Introduction

Liouville (1832) and Riemann (1876) developed logical definitions of fractional operations [4] & [5]. The definite integral, called the Riemann-Liouville integral, for integration of arbitrary order v

$$ {}_{c}D_{x}^{-v} f(x) = \frac{1}{\Gamma(v)} \int_{c}^{x} (x-t)^{v-1} f(t)dt. \qquad \text{- (1.1)} $$

Where $\operatorname{Re}(v) > 0$ to ensure convergence of integral. When c = 0 we have Riemann's definition and when c = -∞ we have Liouville's definition. The function f is such that on [c, x] the integral converges. For values of $v < 0$ the integral (1.1) would, in general, diverge but because ${}_{c}D_{x}^{-v} f(x)$ can be shown to be an analytic function of v in the region of convergence $\operatorname{Re}(v) > 0$, it can be defined outside this region of convergence as the analytic continuation of this function.

The notation ${}_{c}D_{x}^{-v}$, which denotes the operator of integration of arbitrary order and the corresponding operator ${}_{c}D_{x}^{v}$, which denotes differentiation of arbitrary order was introduced by Davis (1936) [2]. The lower subscripts adjoining D are the terminals of integration and are vital in applications to avoid ambiguities. Differentiation of arbitrary order is given by

$$ \begin{aligned} {}_{c}D_{x}^{v} f(x) &= {}_{c}D_{x}^{m-p} f(x) = {}_{c}D_{x}^{m}\, {}_{c}D_{x}^{-p} f(x) \\ &= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(p)} \int_{c}^{x} \left(x-t\right)^{p-1} f(t)dt \right\} \end{aligned} \qquad \text{- (1.2)} $$

where (for convenience) m, is considered the least integer greater than $\operatorname{Re}(v)$, v = m-p, 0<p≤1, and ${}_{c}D_{x}^{m}$ is the ordinary differentiation operator d^m / dx^m .For a wide class of function f, the integral (1.2) is a beta integral and is readily evaluated.

For a function of the type x^a , Re (a) > -1 and Re (v) > 0, we have for integration and differentiation of arbitrary order

$$ {}_0D_x^{-v}x^a = \frac{\Gamma(a+1)x^{a+v}}{\Gamma(a+v+1)} \qquad \text{- (1.3)} $$

$$ {}_0D_x^{v}x^a = \frac{\Gamma(a+1)x^{a-v}}{\Gamma(a-v+1)}. \qquad \text{- (1.4)} $$

2 Some Important Sesquiderivative and Sesquiintegral [1]

1. Definition: - Fractional Integral

Let $q > 0$ denote a real number and f a continuous function. The fractional integral of f of order -q is define as

$$ \frac{d^{-q}f(x)}{dx^{-q}} = \frac{1}{\overline{\lfloor q}}\int_0^x \frac{f(y)}{(x-y)^{-q+1}}\,dy \qquad \text{-(2.1)} $$

the fractional derivative is now defined by applying differentiation a whole number of times to a fractional integral.

2. Definition: - Fractional Derivative

Let q > 0 denote a real number and n the smallest integer exceeding q.The fractional derivative of f of order q is define as

$$ \frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n}\left(\frac{d^{-(n-q)}f(x)}{dx^{-(n-q)}}\right). \qquad \text{- (2.2)} $$

*** Using these definitions we find the sesquiderivative and sesquiintegral of x^2**

Sol. By definition of fractional integral (2.1)

$$\frac{d^{-3/2}x^2}{dx^{-3/2}} = \frac{1}{\Gamma(3/2)}\int_0^x \frac{y^2}{(x-y)^{-\frac{3}{2}+1}}dy$$

$$= \frac{1}{\Gamma(3/2)}\int_0^x \frac{y^2}{(x-y)^{-1/2}}dy \qquad - (2.3)$$

Changing y to $x-\theta,$ (2.3) becomes

$$\frac{d^{-3/2}x^2}{dx^{-3/2}} = \frac{1}{\Gamma(3/2)}\int_0^x \frac{(x-\theta)^2\,d\theta}{\theta^{-1/2}}$$

$$\therefore \frac{d^{-3/2}x^2}{dx^{-3/2}} = \frac{32x^{7/2}}{105\sqrt{\pi}} \qquad -(2.4)$$

Now put n = 3, q = 3/2 and f(x) = x^2 then from (2.2) and using (2.4), we have

$$\frac{d^{3/2}x^2}{dx^{3/2}} = \frac{d^3}{dx^3}\left(\frac{d^{-3/2}x^2}{dx^{-3/2}}\right)$$

$$\frac{d^{3/2}x^2}{dx^{3/2}} = \frac{d^3}{dx^3}\left(\frac{32x^{7/2}}{105\sqrt{\pi}}\right)$$

$$\frac{d^{3/2}x^2}{dx^{3/2}} = 4\sqrt{x/\pi}$$

(ii) Sesquiderivative and Sesquiintegral of lnx

From equation (2.1)

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}} = \frac{1}{\Gamma(3/2)}\int_0^x \frac{\ln y}{(x-y)^{-\frac{3}{2}+1}}dy$$

$$=\frac{1}{\Gamma(3/2)}\int_0^x \frac{\ln y}{(x-y)^{-1/2}}dy \quad -(2.5)$$

Put y=tx and dy=xdt in (2.5)

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}}=\frac{1}{\Gamma(3/2)}\int_0^1 \frac{\ln(tx)x}{(x-tx)^{-1/2}}dt$$

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}}=\frac{x^{3/2}}{\Gamma(3/2)}\int_0^1 (1-t)^{1/2}\{\ln(x)+\ln(t)\}dt$$

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}}=\frac{x^{3/2}}{\Gamma(3/2)}\left[\ln x\int_0^1\sqrt{(1-t)}dt+\int_0^1(1-t)^{1/2}\ln t dt\right]$$

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}}=\frac{x^{3/2}}{\Gamma(3/2)}\left[\frac{2}{3}\ln x+\int_0^1(1-t)^{1/2}\ln t dt\right] \quad -(2.6)$$

But from [3]

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\ln x dx = B(\mu,\nu)\left[\psi(\mu)-\psi(\mu+\nu)\right], \operatorname{Re}\mu,\nu>0.$$

Put μ=1 and ν = 3/2

$$\int_0^1 x^0(1-x)^{1/2}\ln x dx = B(1,\frac{3}{2})\left[\psi(1)-\psi(5/2)\right] \quad -(2.7)$$

We know that by recurrence relation

$$\psi(1+x)=\psi(x)+\frac{1}{x}$$

$$\psi(5/2)=-2\log 2-\gamma+\frac{8}{3}$$

and

$$\psi(1)=-\gamma$$

From (2.6) and (2.7)

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}} = \frac{x^{3/2}}{\Gamma(3/2)}\left[\frac{2}{3}\ln x + \frac{2}{3}\left(2\log 2 - \frac{8}{3}\right)\right]$$

$$\frac{d^{-3/2}\ln x}{dx^{-3/2}} = \frac{4x^{3/2}}{3\sqrt{\pi}}\left[\ln x + 2\log 2 - \frac{8}{3}\right] \qquad \text{-(2.8)}$$

Now putting q = 3/2, n = 3 and f(x) = lnx in equation (2.2) and using (2.8), we get

$$\frac{d^{3/2}\ln x}{dx^{3/2}} = \frac{d^3}{dx^3}\left(\frac{d^{-3/2}\ln x}{dx^{-3/2}}\right)$$

$$\frac{d^{3/2}\ln x}{dx^{3/2}} = \frac{d^3}{dx^3}\left[\frac{4x^{3/2}}{3\sqrt{\pi}}\left(\ln x + 2\log 2 - \frac{8}{3}\right)\right]$$

$$\frac{d^{3/2}\ln x}{dx^{3/2}} = \frac{x^{-3/2}}{\sqrt{\pi}}\left[1 - \left(\frac{\ln x}{2} + \log 2\right)\right]$$

(iii) Sesquiderivative and Sesquiintegral of sinx

From equation (2.1)

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}} = \frac{1}{\Gamma(3/2)}\int_0^x \frac{\sin y}{(x-y)^{-\frac{3}{2}+1}}dy$$

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}} = \frac{1}{\Gamma(3/2)}\int_0^x \frac{\sin y}{(x-y)^{-1/2}}dy$$

on putting $y = x - \theta^2$ and $dy = -2\theta d\theta$

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}}=\frac{1}{\Gamma(3/2)}\int_0^{\sqrt{x}}\frac{\sin(x-\theta^2)(2\theta)}{\theta^{-1}}d\theta$$

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}}=\frac{2}{\Gamma(3/2)}\int_0^{\sqrt{x}}(\sin x\cos\theta^2-\cos x\sin\theta^2)\theta^2 d\theta$$

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}}=\frac{2}{\Gamma(3/2)}\left[sinx\int_0^{\sqrt{x}}\theta^2\cos\theta^2 d\theta-cosx\int_0^{\sqrt{x}}\theta^2\sin\theta^2 d\theta\right] \qquad \text{-(2.9)}$$

Evaluating the integrals, (2.9) becomes

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}}=\frac{2}{\Gamma(3/2)}\left[sinx\left(\frac{\sqrt{x}\sin x}{2}+\frac{\cos x}{2\sqrt{x}}\right)-cosx\left(\frac{\sqrt{x}\cos x}{-2}+\frac{\sin x}{2\sqrt{x}}\right)\right]$$

Hence on simplifying we get

$$\frac{d^{-3/2}\sin x}{dx^{-3/2}}=2\sqrt{\frac{x}{\pi}} \qquad \text{-(2.10)}$$

Put q = 3/2 , n = 3 and f(x) = sinx in equation (2.2) and using (2.10), we get

$$\frac{d^{3/2}\sin x}{dx^{3/2}}=\frac{d^3}{dx^3}\left(\frac{d^{-3/2}}{dx^{-3/2}}\right)$$

$$\frac{d^{3/2}\sin x}{dx^{3/2}}=\frac{d^3}{dx^3}\left(2\sqrt{\frac{x}{\pi}}\right)$$

$$\frac{d^{3/2}\sin x}{dx^{3/2}}=\frac{3x^{-5/2}}{4\sqrt{\pi}}$$

Similarly we can get the fractional derivative and integral of many interesting function we give them in tabular form.

Some important Sesquiderivative

f	$\frac{d^{3/2} f}{dx^{3/2}}$
0	0
c	$\frac{-cx^{-3/2}}{2\sqrt{\pi}}$
x^0	$\frac{-x^{-3/2}}{2\sqrt{\pi}}$
x	$\frac{1}{\sqrt{\pi x}}$
x^2	$4\sqrt{x/\pi}$
$x^{\frac{1}{2}}$	0
$x^{-\frac{1}{2}}$	0
e^x	$\frac{3(5/x-1)}{4\sqrt{\pi}x^{5/2}}$

$\sin(x)$	$\frac{3}{4\sqrt{\pi}x^{5/2}}$
$\cos(x)$	$\frac{-15}{4\sqrt{\pi}x^{7/2}}$
$\ln(x)$	$\frac{x^{-3/2}}{\sqrt{\pi}}\left[1-\left(\frac{\ln x}{2}+\log 2\right)\right]$

Some important Sesquiintegral

f	$\frac{d^{-3/2}f}{dx^{-3/2}}$
0	0
c	$\frac{4cx^{3/2}}{3\sqrt{\pi}}$
x^o	$\frac{4x^{3/2}}{3\sqrt{\pi}}$
x	$\frac{8x^{5/2}}{15\sqrt{\pi}}$

x^2	$\frac{32x^{7/2}}{105\sqrt{\pi}}$
$x^{\frac{1}{2}}$	$\frac{x^2\sqrt{\pi}}{4}$
$x^{-\frac{1}{2}}$	$x\sqrt{\pi}$
e^x	$-2\left(\frac{x+1}{\sqrt{\pi x}}\right)$
$\sin(x)$	$2\sqrt{\frac{x}{\pi}}$
$\cos(x)$	$\frac{2}{\sqrt{\pi x}}$
$ln(x)$	$\frac{4x^{3/2}}{3\sqrt{\pi}}\left[\ln x+2\log 2-\frac{8}{3}\right]$

References

[1] L.Borredon, B. Henry and S.Wearne, Differentiating the non-differentiable - Fractional Calculus. Parabola 35(2), (1999), 1-9.

[2] T.Davis,Harold, The theory of Linear Operators, Principia Press,1936.

[3] I.S.Gradshteyn and I.M.Ryzhik, Tables of Integrals, Series, and Products, Academic Press, New York, 1980.

[4] J.Liouville, Mémoire sur l'intégration de l'équation
$(mx^2 + nx + p)\, d^2y \big/ dx^2 + (qx + r)\, dy \big/ dx + sy = 0$
àl'aide des différentielles à indices quelconques. J.E'cole Ploytechnique, 13(21), (1832), 163-86.

[5] B.Riemann, Versuch einer Auffassung der Integration und Differentiation.Gesammelte Werke, Teubner, 1876 , 331-44.

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