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Bi-Distance Pattern Uniform Number

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Abstract

A graph $G = (V, E)$ is *Bi- Distance Pattern Uniform* if there exists $M \subseteq V(G)$ such that the $M-$distance pattern $f_M(u) = \{d(u, v) : v \in M\}$ is identical for all u in M and $f_M(v)$ is identical for all v in $V - M$. The set M is called *Bi-DPU set*. The least cardinality of Bi-DPU set in G is called the Bi-DPU number of G. In this paper, we initiate a study on Bi-DPU number of different classes of graphs.

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1 Introduction

For all terminology and notation in graph theory, we refer the reader to Chartrand [1]. Unless mentioned otherwise, all graphs considered in this paper are finite, simple and connected.
Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex u in G is associated with the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between the vertices u and v in G, is called the M-distance pattern of u [3]. Germina and Rency [2] defined Bi- Distance Pattern Uniform (Bi-DPU) Graph as follows: Let $G = (V, E)$ be a (p, q) graph

and M be any non-empty subset of $V(G)$. Then, the M-distance pattern of u is the set $f_M(u) = \{d(u,v) : v \in M\}$, where $d(u,v)$ denotes the usual distance between u and v in G. If $f_M(u)$ is identical for all $u \in M$ and $f_M(v)$ is identical for all $v \in V - M$, then G is called a Bi- distance pattern uniform (Bi-DPU) graph. The set M is called the Bi-DPU set. We need the following known results.

Theorem 1.1. *[2] A non self-centered graph G is a Bi-DPU graph if and only if G has exactly two eccentricities and $Cen(G)$ is self-centered.*

Theorem 1.2. *[2] A tree T is a Bi-DPU graph if and only if $T \cong K_{1,n}$ or $B_{m,n}$ where $B_{m,n}$ is a bistar.*

1.1 Bi-DPU numbers of different classes of graphs

Definition 1.3. Bi-DPU number of a graph G, denoted by $\varsigma_B(G)$, is the minimum cardinality of a Bi-DPU set in G.

Theorem 1.4. *A graph G is a Bi-DPU graph with $\varsigma_B(G) = 1$ if and only if G has at least one vertex of full degree.*

Proof. Assume that G has at least one full degree vertex. Let u be a full degree vertex. Choose $M = \{u\}$. Then, $f_M(u) = \{0\}$ and $f_M(v) = \{1\}, \forall v \in V - M$. Hence, $\varsigma_B(G) = 1$. Conversely, let $\varsigma_B(G) = 1$, that is $|M| = 1$. Hence, $f_M(u) = \{0\}$ whenever $u \in M$ and $f_M(v) = \{1\}$ for all $v \neq u$, which implies the vertex in M should necessarily be of full degree. □

Corollary 1.5. *$\varsigma_B(K_n) = 1$, $\varsigma_B(W_n) = 1$ where $W_n = K_1 + C_n$ is the Wheel graph and $\varsigma_B(K_{1,n}) = 1$.*

Corollary 1.6. *Given a fixed natural number p, for a (p,q) graph with Bi-DPU number 1, we have, $p - 1 \leq q \leq \frac{p(p-1)}{2}$.*

Proof. We have, $K_{1,p-1}$ and K_p are Bi-DPU graphs with Bi-DPU number 1 of smallest and largest size respectively, we get $p - 1 \leq q \leq \frac{p(p-1)}{2}$. □

Theorem 1.7. *For bistar $B_{m,n}$, $\varsigma_B(B_{m,n}) = 2$.*

Proof. Let $B_{m,n}$ be a bistar with Bi-DPU set $M = \{u, v\}$, where u and v are central vertices of $B_{m,n}$. That is, $\varsigma_B(B_{m,n}) \leq 2$. But, $B_{m,n}$ has no full degree vertex, $\varsigma_B(B_{m,n}) \neq 1$. Therefore, $\varsigma_B(B_{m,n}) = 2$. □

Theorem 1.8. $\varsigma_B(K_{m,n}) = 2,\ m, n \geq 2.$

Proof. Let $\{X, Y\}$ be the bipartition of the vertex set of $K_{m,n}$. Choose $M = \{u, v\}$ where $u \in X$ and $v \in Y$. Then, $f_M(u) = f_M(v) = \{0, 1\}$ and $f_M(w) = \{1, 2\}$ $\forall w \in V - M$. Hence, $\varsigma_B(K_{m,n}) \leq 2$. Also, $K_{m,n}$ contains no full degree vertex, $\varsigma_B(K_{m,n}) \neq 1$. Hence, $\varsigma_B(K_{m,n}) = 2$. □

Theorem 1.9. $\varsigma_B(C_n) = \begin{cases} \frac{n}{3}, & \textit{if } n \textit{ is a multiple of 3} \\ \frac{n}{2}, & \textit{if } n \textit{ is even and not a multiple of 3} \\ n-1, & \textit{if } n \textit{ is odd and not a multiple of 3} \end{cases}$

Proof. Let C_n be a cycle on n vertices and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$.
Case 1: n is a multiple of 3. Choose $M = \{v_1, v_4, v_7, \ldots, v_{n-2}\}$. Then, for all $v_i \in M$, $f_M(v_i) = \begin{cases} \{0, 3, 6, \ldots, \frac{n}{2}\} & \text{if n is even} \\ \{0, 3, 6, \ldots, \frac{n-3}{2}\} & \text{if n is odd} \end{cases}$
are identical sets.
Also, for all $v_j \in V - M$, $f_M(v_j) = \begin{cases} \{1, 2,\ 4, 5,\ 7, 8\ \ldots, \frac{n-2}{2}\} & \text{if n is even} \\ \{1, 2,\ 4, 5,\ 7, 8\ \ldots, \frac{n-1}{2}\} & \text{if n is odd} \end{cases}$
are identical sets. Hence, $\varsigma_B(C_n) \leq \frac{n}{3}$ if n is a multiple of 3.
Now, we prove that $\varsigma_B(C_n) = \frac{n}{3}$. If possible, choose $M' \subset V(C_n)$ with $|M'| < |M|$. Let $M' = \{u_1, u_2, \ldots, u_l\}$, where $l < \frac{n}{3}$ and each u_j is some $v_i \in V(C_n)$. Then, there exists at least one vertex $u_j \in M'$ such that $d(u_j, u_{j+1}) > 3$. Assume $d(u_j, u_{j+1}) = 4$ say. Let the shortest $u_j - u_{j+1}$ path in C_n be $u_j v_k v_{k+1} v_{k+2} u_{j+1}$. Then, $1 \in f_{M'}(v_k)$ and $1 \notin f_{M'}(v_{k+1})$, M' can not be a Bi-DPU set for C_n. Therefore, $\varsigma_B(C_n) = \frac{n}{3}$ whenever n is a multiple of 3.
Case 2: n is even and not a multiple of 3. Choose $M = \{v_2, v_4, \ldots, v_n\}$, the set of all alternate vertices of C_n.
Then, for all $v_i \in M$,
$f_M(v_i) = \begin{cases} \{0, 2, 4, \ldots, \frac{n}{2}\} & \text{if } n = 2m \text{ and m is even} \\ \{0, 2, 4, \ldots, \frac{n-2}{2}\} & \text{if } n = 2m \text{ and m is odd} \end{cases}$
Therefore, $f_M(v_i)$ are identical sets.
Also, for all $v_j \in V - M$,
$f_M(v_j) = \begin{cases} \{1, 3, 5, \ldots, \frac{n-2}{2}\} & \text{if } n = 2m \text{ and m is even} \\ \{1, 3, 5, \ldots, \frac{n}{2}\} & \text{if } n = 2m \text{ and m is odd} \end{cases}$
Therefore, $f_M(v_j)$ are identical sets. Hence, $\varsigma_B(C_n) \leq \frac{n}{2}$. Choose $M' \subset V(C_n)$ with $|M'| < \frac{n}{2}$ and if possible $|M'| = \frac{n}{2} - 1$. Let $M' = \{u_1, u_2, \ldots, u_l\}$, where each u_j is some $v_i \in V(C_n)$. Then, there exists at least one $u_j \in M'$ such that $d(u_j, u_{j+1}) \geq 3$. In this case, let $d(u_j, u_{j+1}) = 3$, so that $3 \in f_{M'}(u_j)$, $3 \notin f_{M'}(u_{j-1})$ and hence M' is not a Bi-DPU set. Now, when $d(u_j, u_{j+1}) = 4$, the shortest $u_j - u_{j+1}$ path in C_n is $u_j v_k v_{k+1} v_{k+2} u_{j+1}$, so that $1 \in f_{M'}(v_k)$, $1 \notin f_{M'}(v_{k+1})$, M' is not a Bi-DPU set for C_n. A similar argument follows when $d(u_j, u_{j+1}) = 5, 6, \ldots, \frac{n}{2}$. Therefore, in all the cases $|M'|$ is not a Bi-DPU set for C_n. Hence, we conclude that $\varsigma_B(C_n) = \frac{n}{2}$ if n is even and not a multiple of 3.
Case 3: n is odd and not a multiple of 3. Choose M as the set of $n - 1$

vertices of C_n. Then, $f_M(v_i) = \{0, 1, 2, \dots, \frac{n-1}{2}\}$, $\forall v_i \in M$ and $f_M(v_j) = \{1, 2, \dots, \frac{n-1}{2}\}$, $\forall v_j \in V - M$. Hence, $\varsigma_B(C_n) \leq n - 1$. Choose $M' \subset V(C_n)$ with $|M'| < n-1$. If M' is a Bi-DPU set for C_n then there are two possibilities. Either the elements of M' are alternate vertices of $V(C_n)$ or there are two elements of $V-M'$ lies between the any two elements of M'. If the elements of M' are alternate vertices of $V(C_n)$ then $|M'| = |V - M'|$, $|V(C_n)|$ is even, which is not possible. If there are two elements of $V-M'$ lies between the two elements of $|M'|$ then, $|V - M'| = 2|M'|$, $|V(C_n)|$ is a multiple of three, which is not possible. Hence, M' is not a Bi-DPU set for C_n. Therefore, $\varsigma_B(C_n) = n - 1$ whenever n is odd and not a multiple of 3. □

The *shadow graph* $S(G)$ of a graph G is obtained from G by adding, for each vertex v of G, a new vertex v', called the shadow vertex of v, and joining v' to the neighbors of v in G.

Theorem 1.10. *For the shadow graph $S(K_n)$ of complete graph, $\varsigma_B(S(K_n)) = n$.*

Proof. Let the vertices of K_n be $\{v_1, v_2, \dots, v_n\}$ and the corresponding shadow vertices be $\{v'_1, v'_2, \dots, v'_n\}$. Choose $M = \{v_1, v_2, \dots, v_n\}$. Then, $f_M(v_i) = \{0, 1\}$, $\forall v_i \in M$ and $f_M(v'_i) = \{1, 2\}$, $\forall v'_i \in V - M$. Hence, $\varsigma_B(S(K_n)) \leq n$. Choose $M' \subset V(S(K_n))$ such that $|M'| < |M|$.
Case 1: $M' \subset V(K_n)$
Then, there exists at least one $v_i \in V(K_n)$ which does not belong to M' and for $v_i, v'_j \in V - M'$, $f_{M'}(v_i) = \{1\}$ and $f_{M'}(v'_j) = \{1, 2\}$. Hence, M' is not a Bi-DPU set.
Case 2: $M' \subset V(S(K_n)) - V(K_n)$
Then, there exists at least one $v'_i \in V(S(K_n)) - V(K_n)$ which does not belong to M' and for $v_i, v'_i \in V - M'$, $f_{M'}(v_i) = \{1\}$ and $f_{M'}(v'_i) = \{2\}$. Hence, M' is not a Bi-DPU set.
Case 3: M' consists of vertices of K_n, shadow vertices and $|M'| < n$.
Then, there exists $v_i, v'_i \in V - M'$ and since, v_i is adjacent to all vertices of $V(S(K_n))$ except v'_i, $f_{M'}(v_i) = \{1\}$ and $2 \in f_{M'}(v'_i)$. Hence, M' is not a Bi-DPU set. Therefore, $\varsigma_B(S(K_n)) = n$. □

Theorem 1.11. $\varsigma_B(P_m + P_n) = \begin{cases} 4 & \text{if } m, n \geq 4 \\ 1 & \text{otherwise.} \end{cases}$

Proof. Let $G \cong P_m + P_n$; $V(P_n) = \{u_1, u_2, \dots, u_n\}$ *and* $V(P_m) = \{v_1, v_2, \dots, v_m\}$
Case 1: m *or* $n < 4$
If m or n less than 4 then G has at least one full degree vertex. Therefore, $\varsigma_B(P_m + P_n) = 1$.
Case 2: $m, n \geq 4$.

Let $M = \{u_i, u_{i+1}, v_j, v_{j+1}\}$, $u_i \in V(P_n), v_j \in V(P_m)$. Then, $f_M(u) = \{0, 1\}$ for all $u \in M$ and $f_M(v) = \{1, 2\}$ for all $v \in V - M$. Then, $\varsigma_B(G) \leq 4$. Next, we prove that $\varsigma_B(G) \not< 4$. Since, G has no full degree vertex, $\varsigma_B(G) \neq 1$. Also, $\varsigma_B(G) \neq 2$. For,

Subcase 2.1.1: Choose $M = \{u_i, v_j\}$ where $u_i \in V(P_n)$ and $v_j \in V(P_m)$. Then, $f_M(u_{i+1}) = f_M(u_{i-1}) = f_M(v_{j-1}) = f_M(v_{j+1}) = \{1\}$ and for all other vertices in $V - M$, $f_M(v) = \{1, 2\}$. Hence, M is not a Bi-DPU set.

Subcase 2.1.2: Choose $M = \{u_i, u_j\}$. Then, $f_M(v_k) = \{1\}$ for all v_k, $f_M(u_l) = \{1, 2\}$ where u_l is adjacent to u_i or u_j and $f_M(u_r) = \{2\}$ where u_r is not adjacent to both u_i and u_j. Hence, M is not a Bi-DPU set. Now, $\varsigma_B(G) \neq 3$. For,

Subcase 2.2.1: Choose $M = \{u_i, u_j, v_k\}$. Then, $f_M(v_{k+1}) = f_M(v_{k-1}) = \{1\}$ and $f_M(v_s) = \{1, 2\}$ for all $v_s \in V - M$. Hence, M is not a Bi-DPU set.

Subcase 2.2.2: Choose $M = \{u_i, u_j, u_k\}$. Then, $f_M(v_i) = \{1\}$ for all v_i and $f_M(u_s) = \{1, 2\}$ for some $u_s \in V - M$. Hence, M is not a Bi-DPU set.

Therefore, we conclude that $\varsigma_B(P_m + P_n) = 4$. □

Theorem 1.12. *The ladder* $L_n \cong P_n \times P_2$ *is a Bi-DPU graph if and only if* $n \leq 4$ *and* $\varsigma_B(L_n) = \begin{cases} n & \text{if } n = 1, 2, 4 \\ 2 & \text{if } n = 3 \end{cases}$

Proof. First we prove that L_n is a Bi-DPU graph for $n \leq 4$.

Case 1: When $n = 1$, $L_1 \cong K_2$, by theorem 1.4, L_1 is a Bi-DPU graph, $\varsigma_B(L_1) = 1$.

Case 2: When $n = 2$, $L_2 \cong C_4$, by theorem 1.9, L_2 is a Bi-DPU graph, $\varsigma_B(L_2) = 2$.

Case 3: $n = 3$. Let v_1, v_2, v_3 and v_4 be the vertices of L_3 corresponding to the eccentricity 3 and u_1 and u_2 be the vertices of L_3 corresponding to the eccentricity 2. Choose $M = \{u_1, u_2\}$. Then, $f_M(u_1) = f_M(u_2) = \{0, 1\}$ and $f_M(v) = \{1, 2\}$ for all $v \in V - M$. Hence, M is a Bi-DPU set for L_3 and $\varsigma_B(L_3) \leq 2$. Since, L_3 has no full degree vertex, $\varsigma_B(L_3) \neq 1$. Therefore, $\varsigma_B(L_3) = 2$.

Case 4: $n = 4$. Let v_1, v_2, v_3 and v_4 be the vertices of L_4 corresponding to the eccentricity 4 and u_1, u_2, u_3 and u_4 be the vertices of L_4 corresponding to the eccentricity 3. Choose $M = \{u_1, u_2, u_3, u_4\}$. Then, $f_M(u_i) = \{0, 1, 2\}$ for $i = 1, 2, 3, 4$ and $f_M(v_j) = \{1, 2, 3\}$ for $j = 1, 2, 3, 4$. Therefore, M is a Bi-DPU set for L_4 and $\varsigma_B(L_4) \leq 4$. We prove $\varsigma_B(L_4) \not< 4$. Since, L_4 contains no full degree vertex, $\varsigma_B(L_4) \neq 1$. Also, $\varsigma_B(L_4) \neq 2$. For,

Subcase 4.1.1: Choose $M = \{v_i, u_j\}$. Then, there exists $v_k \in V - M$ such that $4 \in f_M(v_k)$ and $4 \notin f_M(v)$ for all $v \in (V - M) - \{v_k\}$. Hence, M is not a Bi-DPU set for L_4.

Subcase 4.1.2: Choose $M = \{v_i, v_j\}$. If $d(v_i, v_j) = 1$ *or* 3 then $4 \in f_M(v_k)$ for all $v_k \in V - M$ and $4 \notin f_M(u_i)$ for all $u_i \in V - M$. If $d(v_i, v_j) = 4$ then

$f_M(u_1) = f_M(u_3) = \{1, 3\}$ and $f_M(u_2) = f_M(u_4) = \{2\}$. Hence, M is not a Bi-DPU set for L_4.
Subcase 4.1.3: Choose $M = \{u_i, u_j\}$. Then, there are two vertices $v_l, v_k \in V - M$ such that $3 \in f_M(v_l), f_M(v_k)$ and $3 \notin f_M(v)$ for all $v \in (V - M) - \{v_l, v_k\}$. Hence, M is not a Bi-DPU set for L_4.
Now, $\varsigma_B(L_4) \neq 3$. For,
Subcase 4.2.1: Choose $M = \{u_i, u_j, v_k\}$. If any two vertices in M are adjacent then $f_M(u)$ is not identical for all $u \in M$. If all the vertices in M are non-adjacent then there exists a vertex $v_s \in V - M$ such that $4 \in f_M(v_s)$ and $4 \notin f_M(v)$ for all $v \in (V - M) - \{v_s\}$. Hence, M is not a Bi-DPU set for L_4.
Subcase 4.2.2: Choose $M = \{v_i, v_j, u_k\}$. If v_i is adjacent to both v_j and u_k then $f_M(v_i) = \{0, 1\}$ and $f_M(v_j) = f_M(u_k) = \{0, 1, 2\}$. If any two vertices in M are adjacent then $f_M(v_i) \neq f_M(v_j) \neq f_M(u_k)$. If no two elements in M are adjacent and $d(v_i, v_j) = 3$ then $4 \in f_M(v_s), f_M(v_k)$ and $4 \notin f_M(v)$ for all $v \in (V - M) - \{v_s, v_k\}$. If no two elements in M are adjacent and $d(v_i, v_j) = 4$ then $4 \in f_M(v_i), f_M(v_j)$ and $4 \notin f_M(u_k)$. Hence, M is not a Bi-DPU set for L_4.
Therefore, we conclude that $\varsigma_B(L_4) = 4$.
Conversely, assume that L_n is a Bi-DPU graph. We have to prove that L_n is a Bi-DPU graph only for $n \leq 4$. If possible suppose $n \geq 5$. Then, L_n has more than two eccentricities. Hence, by theorem 1.4, L_n is not a Bi-DPU graph. Therefore, L_n is a Bi-DPU graph for $n \leq 4$. □

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