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On Additive Maps Preserving the Hyper-range or Hyper-kernel of Operators

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Abstract

Let X and Y be two infinite dimensional Banach spaces. We determine the forms of surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ which preserve the hyper-range's codimension or the hyper-kernel's dimension. We also show that if $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the hyper-range or the hyper-kernel, then there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.

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1 Introduction

Let X and Y be two infinite dimensional Banach spaces, and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, we write T^* for its adjoint, $\mathrm{N}(T)$ for its kernel, $\mathrm{R}(T)$ for its range. The *hyper-kernel* and the *hyper-range* of T are defined respectively by

$$\mathcal{N}^\infty(T) := \bigcup_{n\in\mathbb{N}} \mathrm{N}(T^n) \text{ and } \mathcal{R}^\infty(T) := \bigcap_{n\in\mathbb{N}} \mathrm{R}(T^n).$$

A surjective additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is said to *preserve the hyper-kernel's dimension* if $\dim \mathcal{N}^\infty(\phi(T)) = \dim \mathcal{N}^\infty(T)$ for all $T \in \mathcal{L}(X)$ and is said, in the case $X = Y$, to *preserve the hyper-kernel* if $\mathcal{N}^\infty(\phi(T)) = \mathcal{N}^\infty(T)$ for all $T \in \mathcal{L}(X)$. Clearly, every such map preserves the set of injective operators in both directions.

A surjective additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is said to *preserve the hyper-range's codimension* if $\operatorname{codim} \mathcal{R}^\infty(\phi(T)) = \operatorname{codim} \mathcal{R}^\infty(T)$ for all $T \in \mathcal{L}(X)$ and is said, in the case $X = Y$, to *preserve the hyper-range* if $\mathcal{R}^\infty(\phi(T)) = \mathcal{R}^\infty(T)$ for all $T \in \mathcal{L}(X)$. Analogously, every such map preserves the set of surjective operators in both directions.

For more details on additive or linear maps that preserve injective operators, or surjective operators, or parts of spectrum, we refer the reader to some survey articles [2, 3, 10].

In [9], the forms of all surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ preserving the kernel's dimension or the range's codimension are determined, and it is established also that $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the kernel (respectively, the range) if and only if there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = AT$ (respectively, TA) for all $T \in \mathcal{L}(X)$.

In this paper, we determine the forms of all surjective additive maps, not necessarily unital, $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ that preserve the hyper-kernel's dimension or the hyper-range's codimension, and consequently we obtain that if $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the hyper-kernel or the hyper-range, then $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$, where μ is a nonzero scalar in $\mathbb{C}$.

2 Preliminaries and Notations

Let τ be a ring automorphism of $\mathbb{C}$. An additive map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is called *τ-quasilinear* if $\phi(\lambda T) = \tau(\lambda)\phi(T)$ for all $\lambda \in \mathbb{C}$ and $T \in \mathcal{L}(X)$. If $\tau(\lambda) = \overline{\lambda}$, then ϕ is called *conjugate linear*. We say that ϕ is unital if $\phi(I) = I$, where I stands for the unit of both $\mathcal{L}(X)$ and $\mathcal{L}(Y)$.

Let x be a nonzero vector in X and f be a nonzero linear form in the topological dual X^* of X. We denote, as usual, by $x \otimes f$ the rank one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. Note that $x \otimes f$ is a projection if and only if $f(x) = 1$, and it is nilpotent if and only if $f(x) = 0$. The adjoint of such operator is given by $(x \otimes f)^* = f \otimes Jx$, where J is the natural embedding of X to X^{**}.

For $T \in \mathcal{L}(X)$, the surjectivity spectrum $\sigma_s(T)$ and the approximate point spectrum $\sigma_{ap}(T)$ are defined by $\sigma_s(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective }\}$ and $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective or its range is not closed }\}$ respectively.

For main results, we need the following lemmas.

Lemma 2.1. *Let x be a nonzero vector in X and f be a nonzero functional in X^*. Then the following statements hold:*

(i) *if $f(x) \neq 0$ then $\mathcal{R}^\infty(x \otimes f) = \mathrm{R}(x \otimes f) = \mathrm{span}\,\{x\}$;*

(ii) *if $f(x) = 0$ then $\mathcal{R}^\infty(x \otimes f) = \{0\}$;*

(iii) *if $f(x) = 1$ then*

$$\mathrm{R}(I - x\otimes f) = \mathrm{N}(x\otimes f) = \mathrm{N}(f) \text{ and } \mathrm{N}(I - x\otimes f) = \mathrm{R}(x\otimes f) = \mathrm{span}\,\{x\}$$

(iv) $(x \otimes f)\mathcal{R}^\infty(x \otimes f) = \mathcal{R}^\infty(x \otimes f)$.

Proof. It is easy to verify that $(x\otimes f)^n = (f(x))^{n-1}x\otimes f$ for all integer $n \geq 2$, and so (i) and (ii) hold.
If $f(x) = 1$ then $x \otimes f$ is a projection, and clearly we get (iii).
The statement (iv) is immediate by part (i) and (ii). □

Lemma 2.2. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map that preserves surjective operators, or injective operators, in both directions then $\phi(I)$ is invertible.*

Proof. Suppose that ϕ preserves surjective operators, or injective operators, in both directions. From Claim 2 in proof of [2, Theorem 3.1] it follows that ϕ preserves in both directions the set of operators of rank one, and consequently it takes one of the following forms:

$$(1) \quad \phi(x \otimes f) = Gx \otimes Hf \quad \text{for all } \ x \in X \ \text{ and } \ f \in X^*,$$

or

$$(2) \quad \phi(x \otimes f) = Kf \otimes Lx \quad \text{for all } \ x \in X \ \text{ and } \ f \in X^*,$$

where $G : X \to Y$, $H : X^* \to Y^*$, $K : X^* \to Y$ and $L : X \to Y^*$ are τ-quasilinear bijective maps, and $\tau : \mathbb{C} \to \mathbb{C}$ is a ring automorphism, see [8]. Thus we proceed as in the proof of [9, Lemma 7]. □

Given $T \in \mathcal{L}(X)$, we denote by $\mathrm{K}(T)$ the analytic core of T, defined as the set of all $x \in X$ for which there exist $a > 0$ and a sequence (x_n) in X satisfying:

(i) $x_0 = x, Tx_{n+1} = x_n$ and

(ii) $\| x_n \| \leq a^n \| x \|$, for all $n \geq 1$.

The following lemma describes some basic properties of $\mathrm{K}(T)$; see [1, 7].

Lemma 2.3. *Let $T \in \mathcal{L}(X)$. Then:*

(i) $\mathrm{K}(T)$ *is a (not necessarily closed) subspace of* X*;*

(ii) $T\mathrm{K}(T) = \mathrm{K}(T)$*;*

(iii) *if* M *is a closed subspace of* X *and* $TM = M$ *then* $M \subseteq \mathrm{K}(T)$*;*

(iv) $\mathrm{K}(T) \subseteq \mathcal{R}^\infty(T)$*.*

Lemma 2.4. *Let* $T \in \mathcal{L}(X)$*. If* T *is a projection or of rank one then* $\mathrm{K}(T) = \mathcal{R}^\infty(T)$*.*

Proof. Let $T \in \mathcal{L}(X)$. If T is a projection then, obviously, $T\mathcal{R}^\infty(T) = \mathcal{R}^\infty(T)$ holds. Now if T is of rank one then, by Lemma 2.1 (iv) we have also, $T\mathcal{R}^\infty(T) = \mathcal{R}^\infty(T)$.

Therefore, Lemma 2.3 (iii), (iv) completes the proof . □

3 Main Results

Theorem 3.1. *Let* $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ *be a surjective additive map that preserves the hyper-range's codimension then :*

(i) *either there exist a nonzero scalar* $\mu \in \mathbb{C}$ *and an invertible bounded linear, or conjugate linear, operator* $A : X \to Y$ *such that* $\phi(T) = \mu ATA^{-1}$ *for all* $T \in \mathcal{L}(X)$*; or*

(ii) *there exist a nonzero scalar* $\mu \in \mathbb{C}$ *and an invertible bounded linear, or conjugate linear, operator* $A : X^* \to Y$ *such that* $\phi(T) = \mu AT^*A^{-1}$ *for all* $T \in \mathcal{L}(X)$*. In this case* X *and* Y *are reflexive.*

Proof. Suppose that ϕ preserves the hyper-range's codimension, then it preserves surjective operators in both directions, and so by Lemma 2.2, $\phi(I)$ is invertible. Let $\psi(T) = S^{-1}\phi(T)$ for all $T \in \mathcal{L}(X)$, where $S = \phi(I)$. Then ψ is a unital map that preserves surjective operators in both directions, and consequently, it follows from [2] that :

(i) either there exists an invertible bounded linear, or conjugate linear, operator $A : X \to Y$ such that $\psi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(X)$, or

(ii) there exists an invertible bounded linear, or conjugate linear, operator $A : X^* \to Y$ such that $\psi(T) = AT^*A^{-1}$ for all $T \in \mathcal{L}(X)$.

Suppose that ψ takes the first form (i). Let $y \in Y$ and $g \in Y^*$ be such that $g(y) = 1$. Since ψ is surjective, there exists $T \in \mathcal{L}(X)$ with $\psi(T) = I - y \otimes g$. Then we have $T = A^{-1}(I - y \otimes g)A = I - A^{-1}y \otimes A^*g$. Clearly, T is a projection, and so $\mathcal{R}^\infty(T) = \mathrm{R}(T) = \mathrm{N}(A^{-1}y \otimes A^*g) = \mathrm{N}(A^*g)$. Hence, it follows that

$\operatorname{codim} \mathcal{R}^\infty(\phi(T)) = \operatorname{codim} \mathcal{R}^\infty(T) = 1$, and since $\mathcal{R}^\infty(\phi(T)) \subseteq \mathrm{R}(\phi(T))$, we get that $\mathcal{R}^\infty(\phi(T)) = \mathrm{R}(\phi(T))$. In particular we have

$$\mathrm{R}(S - Sy \otimes g) = \mathrm{R}((S - Sy \otimes g)^2).$$

Let $u \in Y$ be such that $(S - Sy \otimes g)Sy = (S - Sy \otimes g)^2 u$. Applying S^{-2} we obtain

$$\begin{aligned} y - g(Sy)S^{-1}y &= (S^{-1} - S^{-1}y \otimes g)(Su - g(u)Sy) \\ &= u - g(u)y - g(Su - g(u)Sy)S^{-1}y \end{aligned}$$

Applying g we obtain:

$$g(y) - g(Sy)g(S^{-1}y) = g(u) - g(u)g(y) - g(Su - g(u)Sy)g(S^{-1}y).$$

Therefore

$$(g(Sy) - g(Su - g(u)Sy))g(S^{-1}y) = 1$$

which implies that $g(S^{-1}y) \neq 0$. Consequently, y and $S^{-1}y$ are linearly dependent. Hence $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$.

Now suppose that ψ takes the second form; i.e., $\psi(T) = AT^*A^{-1}$ for all $T \in \mathcal{L}(X)$.

It is well known that there exists a reflexive Banach space X with the property T is injective and its range is closed if and only if T is surjective, see [4, 5, 6].

If $\sigma_{ap}(T) \neq \sigma_{su}(T)$ for some $T \in \mathcal{L}(X)$, then there exists a surjective operator $T \in \mathcal{L}(X)$ which is not injective. Therefore T^* is not surjective, and so $\phi(T)$ is also not surjective. We have then $\operatorname{codim} \mathcal{R}^\infty(T) = 0$ and $\operatorname{codim} \mathcal{R}^\infty(\phi(T)) \geq 1$, consequently $\operatorname{codim} \mathcal{R}^\infty(T) \neq \operatorname{codim} \mathcal{R}^\infty(\phi(T))$, a contradiction. Then $\sigma_{ap}(T) = \sigma_{su}(T)$ for all $T \in \mathcal{L}(X)$. It is easy to verify that $\operatorname{codim} \mathrm{R}(T) = \operatorname{codim} \mathrm{R}(T^*)$ for all $T \in \mathcal{L}(X)$. Consequently $\operatorname{codim} \mathcal{R}^\infty(T) = \operatorname{codim} \mathcal{R}^\infty(T^*)$ for all $T \in \mathcal{L}(X)$. Thus we can prove that $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$ as in the first form. □

Theorem 3.2. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then the following assertions are equivalent:*

(i) *ϕ preserves the hyper-range;*

(ii) *there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.*

Proof. Suppose that ϕ preserves the hyper-range, then it preserves hyper-range's codimension. Using Theorem 3.1 we obtain that:

(i) either there exist a nonzero scalar $\mu \in \mathbb{C}$ and an invertible bounded linear, or conjugate linear, operator $A : X \to X$ such that $\phi(T) = \mu A T A^{-1}$ for all $T \in \mathcal{L}(X)$, or

(ii) there exist a nonzero scalar $\mu \in \mathbb{C}$ and an invertible bounded linear, or conjugate linear, operator $A : X^* \to X$ such that $\phi(T) = \mu A T^* A^{-1}$ for all $T \in \mathcal{L}(X)$.

The case (ii) cannot occur. Indeed, assume that ϕ takes the second form. Let $T = x \otimes f$ such that x and Af are linearly independent and $f(x) \neq 0$. We have $\phi(T) = \mu A(f \otimes Jx)A^{-1} = (\mu Af) \otimes (A^{-1})^* Jx$, then by Lemma 1.1, $\mathcal{R}^\infty(\phi(T)) = \operatorname{span}\{Af\}$.

Thus

$$\operatorname{span}\{x\} = \mathcal{R}^\infty(T) = \mathcal{R}^\infty(\phi(T)) = \operatorname{span}\{Af\}.$$

Consequently Af and x are linearly dependent, a contradiction.

Now suppose that ϕ is of the form (i) i.e.; $\phi(T) = \mu A T A^{-1}$ for all $T \in \mathcal{L}(X)$.
Let $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$. Since $\phi(x \otimes f) = \mu A(x \otimes f)A^{-1} = (\mu Ax) \otimes (A^{-1})^* f$, then we get that $\mathcal{R}^\infty(\phi(x \otimes f)) = \operatorname{span}\{Ax\}$.
Thus

$$\operatorname{span}\{x\} = \mathcal{R}^\infty(x \otimes f) = \mathcal{R}^\infty(\phi(x \otimes f)) = \operatorname{span}\{Ax\}.$$

Hence x and Ax are linearly dependent for every $x \in X$. Consequently $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$. □

Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map. The map ϕ is said to *preserve the analytic core's codimension* if $\operatorname{codim} \mathrm{K}(T) = \operatorname{codim} \mathrm{K}(\phi(T))$ for all $T \in \mathcal{L}(X)$. In the case $X = Y$, the map ϕ is said to *preserve the analytic core* if $\mathrm{K}(T) = \mathrm{K}(\phi(T))$ for all $T \in \mathcal{L}(X)$.

As an immediate consequence of the Lemma 2.3, T is surjective if and only if K(T) = X. Thus, it is easy to see that ϕ preserves surjective operators in both directions if it preserves analytic core's codimension or analytic core.

The following characterization of surjective additive maps preserving the analytic core's codimension follows from Lemma 2.4 and the proof of Theorem 3.1.

Theorem 3.3. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map. If ϕ preserves the analytic core's codimension then :*

(i) *either there exist a nonzero scalar $\mu \in \mathbb{C}$ and an invertible bounded linear, or conjugate linear, operator $A : X \to Y$ such that $\phi(T) = \mu A T A^{-1}$ for all $T \in \mathcal{L}(X)$; or*

(ii) *there exist a nonzero scalar* $\mu \in \mathbb{C}$ *and an invertible bounded linear, or conjugate linear, operator* $A : X^* \to Y$ *such that* $\phi(T) = \mu A T^* A^{-1}$ *for all* $T \in \mathcal{L}(X)$. *In this case,* X *and* Y *are reflexive.*

Theorem 3.4. *Let* $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ *be a surjective additive map. Then the following assertions are equivalent:*

(i) ϕ *preserves the analytic core;*

(ii) *there exists a nonzero scalar* $\mu \in \mathbb{C}$ *such that* $\phi(T) = \mu T$ *for all* $T \in \mathcal{L}(X)$.

Proof. Suppose that ϕ preserves the analytic core, then it preserves the analytic core's codimension. By Theorem 3.3, Lemma 2.3 (iv), we can proceed as in the proof of Theorem 3.2. □

Theorem 3.5. *Let* $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ *be a surjective additive map that preserves the hyper-kernel's dimension. Then there exist a nonzero scalar* $\mu \in \mathbb{C}$ *and an invertible bounded linear, or conjugate linear, operator* $A : X \to Y$ *such that* $\phi(T) = \mu A T A^{-1}$ *for all* $T \in \mathcal{L}(X)$.

Proof. Suppose that ϕ preserves the hyper-kernel's dimension. Since T is injective if and only if $\dim \mathcal{N}^\infty(T) = 0$, then ϕ preserves injective operators in both directions. So by Lemma 2.2, we have $\phi(I)$ is invertible. Then the unital map $\psi(T) = (\phi(I))^{-1}\phi(T)$ preserves injective operators in both directions, so there exists an invertible bounded linear, or conjugate linear, operator $A : X \to Y$ such that $\psi(T) = ATA^{-1}$, see [2]. Hence we have $\phi(T) = SATA^{-1}$ where $S = \phi(I)$.

Let $y \in Y$ and $g \in Y^*$ such that $g(y) = 1$. Since ψ is surjective, there exists $T \in \mathcal{L}(X)$ such that $\psi(T) = I - y \otimes g$. Therefore $T = A^{-1}\psi(T)A = I - A^{-1}y \otimes A^*g$, and so T is a projection.
Consequently, $\mathcal{N}^\infty(T) = \operatorname{span}\{A^{-1}y\}$ and $\dim \mathcal{N}^\infty(\phi(T)) = \dim \mathcal{N}^\infty(T) = 1$. Hence $\mathcal{N}^\infty(\phi(T)) = \mathrm{N}(\phi(T))$. In particular $\mathrm{N}(\phi(T)^2) = \mathrm{N}(\phi(T))$.
So we have

$$\mathrm{N}(S(I - y \otimes g)(S - Sy \otimes g)) = \mathrm{N}(S(I - y \otimes g)),$$

then

$$\mathrm{N}((I - y \otimes g)(S - Sy \otimes g)) = \mathrm{N}(I - y \otimes g),$$

thus

$$\mathrm{N}((I - y \otimes g)(S - Sy \otimes g)) = \operatorname{span}\{y\}.$$

Assume that $g(S^{-1}y) = 0$, then $S^{-1}y \in \mathrm{N}((I - y \otimes g)(S - Sy \otimes g))$.
Indeed,

$$\begin{aligned}(I - y \otimes g)(S - Sy \otimes g)S^{-1}y &= (I - y \otimes g)(y - g(S^{-1}y)Sy) \\ &= (I - y \otimes g)y \\ &= 0.\end{aligned}$$

Thus $S^{-1}y \in \text{span}\{y\}$, and so $y = \alpha S^{-1}y$ for some nonzero scalar $\alpha \in \mathbb{C}$. This implies that $g(y) = \alpha g(S^{-1}y) = 0$, a contradiction. Hence $g(S^{-1}y) \neq 0$. Consequently, y and $S^{-1}y$ are linearly dependent . Therefore $S = \mu I$ for some nonzero scalar $\mu \in \mathbb{C}$. And we obtain finally $\phi(T) = \mu ATA^{-1}$ for all $T \in \mathcal{L}(X)$. □

Using the Theorem 3.5, we can get the following theorem.

Theorem 3.6. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then the following assertions are equivalent:*

(i) *ϕ preserves the hyper-kernel;*

(ii) *there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.*

Proof. Suppose that ϕ preserves the hyper-kernel, then it preserves the hyper-kernel's dimension and there exists an invertible bounded linear, or conjugate linear, operator $A : X \to X$ and a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu ATA^{-1}$ for all $T \in \mathcal{L}(X)$.

Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$. Set $T = I - x \otimes f$. Then T is a projection and we have $\mathcal{N}^\infty(T) = \text{span}\{x\}$.

On the other hand, we have $\phi(I - x \otimes f) = \mu A(I - x \otimes f)A^{-1} = \mu(I - Ax \otimes (A^{-1})^* f)$. It is clear that $I - Ax \otimes (A^{-1})^* f$ is a projection, then we get that $\mathcal{N}^\infty(\phi(T)) = \text{span}\{Ax\}$.
Therefore,

$$\text{span}\{x\} = \mathcal{N}^\infty(T) = \mathcal{N}^\infty(\phi(T)) = \text{span}\{Ax\}.$$

Thus $A = \alpha I$ for some nonzero scalar $\alpha \in \mathbb{C}$. Consequently, $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$. □

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