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The Natural Lift Curve of the Spherical Indicatrix of a Null Curve in Minkowski 3-Space

Evren ERGÜN

Ondokuz Mayıs University, Faculty of Arts and Sciences
Department of Mathematics, Samsun, Turkey
eergun@omu.edu.tr

Mustafa ÇALIŞKAN

Gazi University, Faculty of Sciences
Department of Mathematics, Ankara, Turkey
mustafacaliskan@gazi.edu.tr

Abstract

In this study, we dealt with the natural lift curves of the spherical indicatrices of a null curve.Furthermore, some interesting results about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ and $T\left(\Lambda\right)$.

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1 Introduction and Preliminary Notes

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g\left(X,X\right)=-x_1^2+x_2^2+x_3^2,$$

where $X=(x_1,x_2,x_3)\in\mathbb{R}^3$.

A vector $X=(x_1,x_2,x_3)\in\mathbb{R}_1^3$ is said to be timelike if $g\left(X,X\right)<0$, spacelike if $g\left(X,X\right)>0$ and lightlike (or null) if $g\left(X,X\right)=0$. Similarly, an arbitrary curve $\alpha=\alpha\left(t\right)$ in $\mathbb{R}_1^3$ where t is a pseudo-arclength parameter, can be locally timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\alpha}\left(t\right)$ are respectively timelike, spacelike or null (lightlike), for every $t\in I\subset\mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1>0$ (*resp.* $x_1<0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1>0$ (*resp.* $x_1<0$).

The norm of a vector X is defined by [1]

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.$$

The Lorentzian sphere of radius 1 in $\mathbb{R}^3_1$ is given by [1]

$$S^2_1 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 \ : \ g(X,X) = 1 \right\}.$$

The lightlike (null) cone in $\mathbb{R}^3_1$ is given by [1]

$$\Lambda = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}^3_1 - \{0\} : \ g(X,X) = 0 \right\}.$$

We denote the moving Frenet frame along the curve α by $\{T(t), N(t), B(t)\}$, where T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

(i) Let α be a unit speed timelike space curve with curvature κ and torsion τ and Frenet vector fields of α be $\{T, N, B\}$. In this trihedron, T is a timelike vector field, N and B are spacelike vector fields. Then, Frenet formulas are given by [4]

$$\dot{T} = \kappa N, \ \dot{N} = \kappa T + \tau B, \ \dot{B} = -\tau N.$$

(ii) Let α be a unit speed spacelike space curve with a spacelike binormal. For the Frenet vector fields we assume that T and B are spacelike vector fields and N is a timelike vector field .Then, Frenet formulas are given by [4]

$$\dot{T} = \kappa N, \dot{N} = \kappa T + \tau B, \ \dot{B} = \tau N.$$

(iii) Let α be a unit speed spacelike space curve with a timelike binormal. We assume that T and N are spacelike vector fields and B is a timelike vector field. Then, Frenet formulas are given by [4]

$$\dot{T} = \kappa N, \dot{N} = -\kappa T + \tau B, \ \dot{B} = \tau N.$$

(**iv**)Let α be a unit speed null space curve. We assume that T and B are null vector fields and N is a spacelike vector field. Then, Frenet formulas are given by [4]

$$\dot{T} = \kappa N, \dot{N} = \tau T - \kappa B, \dot{B} = -\tau N.$$

Theorem 1 *Let α be a unit speed null space curve. Then, we have*

(1) $\kappa = 0$ *if and only if* α *is a part of a null straight line,*
(2) $\kappa = 1$ *and* $\tau = 0$ *if and only if* α *is a part of the null cubic,*

$$\alpha(s) = \frac{1}{6\sqrt{2}}\left(6s + s^3, 3\sqrt{2}s^2, 6s - s^3\right),$$

(3) $\kappa = 1$ *and* $\tau > 0$ *if and only if* α *is a part of a null circular helix,*

$$\alpha(s) = \frac{1}{K^2}\left(Ks, \kappa\cos(Ks), \kappa\sin(Ks)\right) \text{ with } K = \sqrt{2\tau},$$

(4) $\kappa = 1$ *and* $\tau < 0$ *if and only if* α *is a part of a null hyperbolic helix,*

$$\alpha(s) = \frac{1}{K^2}\left(Ks, \kappa\sinh(Ks), \kappa\cosh(Ks)\right) \text{ with } K = \sqrt{-2\tau}.[4]$$

Definition 1 *Let M be a hypersurface in $\mathbb{R}^3_1$ and $\alpha : I \longrightarrow M$ be a parametrized curve. α is called an integral curve of X if*

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \ \ (\textit{for all } t \in I),$$

where X is a smooth tangent vector field on M, [4]. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M),$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

Definition 2 *For any parametrized curve $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ given by*

$$\overline{\alpha}(t) = \left(\alpha(t), \dot{\alpha}(t)\right) = \dot{\alpha}(t)\,|_{\alpha(t)}$$

is called the natural lift of α on TM. Thus, we can write

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)\,|_{\alpha(t)}\right) = D_{\dot{\alpha}(t)}\dot{\alpha}(t),$$

where D is the Levi-Civita connection on $\mathbb{R}^3_1$, [2].

Definition 3 *A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$*

$$X(V) = +\varepsilon g(S(V), V)N,$$

where $\varepsilon = g(N, N)$, [2].

Theorem 2 *The natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M, [2].*

2 The Natural Lift Curve of The Spherical Indicatrix of a Null Curve in Minkowski 3-Space

Let D, $\bar{D}$ and $\overset{\smile}{\bar{\bar{D}}}$ be Levi-Civita connections on $\mathbb{R}^3_1$, S^2_1 and Λ respectively and ξ be a unit normal vector field of S^2_1 and Λ. Then Gauss Equations are given by the followings

$$D_X Y = \bar{D}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \quad D_X Y = \overset{\smile}{\bar{\bar{D}}}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$

where $\varepsilon = g\left(\xi, \xi\right)$ and S is the shape operator of S^2_1 and Λ and

$$S = I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

2.1.The natural lift of the spherical indicatrix of a unit speed null space curve $\boldsymbol{\alpha}$.

2.1.1. The natural lift of the spherical indicatrix of the tangent vectors of $\boldsymbol{\alpha}$

Let α_T be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_T$ be the natural lift of the curve α_T. If $\overline{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem 2 we have

$$\overset{\smile}{\bar{\bar{D}}}_{\dot{\alpha_T}}\dot{\alpha_T} = 0$$

that is

$$D_{\dot{\alpha_T}}\dot{\alpha_T} = \overset{\smile}{\bar{\bar{D}}}_{\dot{\alpha_T}}\dot{\alpha_T} + \varepsilon g\left(S\left(\dot{\alpha_T}\right), \dot{\alpha_T}\right)\xi$$

$$D_{\dot{\alpha_T}}\dot{\alpha_T} = \varepsilon g\left(S\left(\dot{\alpha_T}\right), \dot{\alpha_T}\right)T$$

where $\varepsilon = g\left(\xi, \xi\right)$ and $\xi = T$.

Since T, N, B are linearly independent, we have $\left(\kappa = 0 \text{ and } \tau = 0\right)$ or $\left(\kappa = 0 \text{ and } \tau \neq 0\right)$.

Corollary 3 *If the natural lift $\overline{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(\Lambda\right)$, then the curve α is a part of a null straight line.*

Corollary 4 *If the natural lift $\overline{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T(\Lambda)$, then we have $\kappa = 0$, $\tau \neq 0$. Therefore there is no such curve satisfying this condition.*

Corollary 5 *If the natural lift $\overline{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T(\Lambda)$, then α_T is a geodesic on $\mathbb{R}^3_1$.*

2.1.2. The natural lift of the spherical indicatrix of the principal normal vectors of $\boldsymbol{\alpha}$

Let α_N be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_N$ be the natural lift of the curve α_N. If $\overline{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem 2 we have

$$\bar{D}_{\dot{\alpha_N}}\dot{\alpha_N} = 0$$

that is

$$D_{\dot{\alpha_N}}\dot{\alpha_N} = \bar{D}_{\dot{\alpha_N}}\dot{\alpha_N} + \varepsilon g\left(S\left(\dot{\alpha_N}\right), \dot{\alpha_N}\right)\xi$$

$$D_{\dot{\alpha_N}}\dot{\alpha_N} = \varepsilon g\left(S\left(\dot{\alpha_N}\right), \dot{\alpha_N}\right)N$$

where $\varepsilon = g(\xi, \xi)$ and $\xi = N$.

Since T, N, B are linearly independent, we have $(\kappa = 0 \text{ and } \tau = 0)$ or $(\kappa = constant \text{ and } \tau = 0)$ or $(\kappa = 0 \text{ and } \tau = constant)$.

Corollary 6 *If the natural lift $\overline{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T(S^2_1)$, then the curve α can be classified as*

(i) $\kappa = 0$ *if and only if α is a part of a null straight line,*
(ii) $\kappa = 1$ *and* $\tau = 0$ *if and only if α is a part of the null cubic,*

$$\alpha(s) = \frac{1}{6\sqrt{2}}\left(6s + s^3, 3\sqrt{2}s^2, 6s - s^3\right),$$

Corollary 7 *If the natural lift $\overline{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T(S^2_1)$, then we have $\kappa = 0$, $\tau \neq 0$. Therefore there is no such curve satisfying this condition.*

Corollary 8 *If the natural lift $\overline{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T(S^2_1)$, then α_N is a geodesic on $\mathbb{R}^3_1$.*

2.1.3. The natural lift of the spherical indicatrix of the binormal vectors of α

Let α_B be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_B$ be the natural lift of the curve α_B. If $\overline{\alpha}_B$ is an integral curve of the geodesic spray, then by using Theorem 2 we have

$$\overset{\asymp}{\bar{D}}_{\dot{\alpha_B}}\dot{\alpha_B} = 0$$

that is

$$D_{\dot{\alpha_B}}\dot{\alpha_B} = \overset{\asymp}{\bar{D}}_{\dot{\alpha_B}}\dot{\alpha_B} + \varepsilon g\left(S\left(\dot{\alpha_B}\right), \dot{\alpha_B}\right)\xi$$

$$D_{\dot{\alpha_B}}\dot{\alpha_B} = \varepsilon g\left(S\left(\dot{\alpha_B}\right), \dot{\alpha_B}\right)B$$

where $\varepsilon = g(\xi, \xi)$ and $\xi = B$. Because T, N, B are linearly independent, we have $(\kappa = 0$ and $\tau = 0)$ or $(\kappa \neq 0$ and $\tau = 0)$.

Corollary 9 *If the natural lift $\overline{\alpha}_B$ of α_B is an integral curve of the geodesic spray on the tangent bundle $T(\Lambda)$, then the curve α can be classified as*

(i) $\kappa = 0$ *if and only if* α *is a part of a null straight line,*
(ii) $\kappa = 1$ *and* $\tau = 0$ *if and only if* α *is a part of the null cubic,*

$$\alpha(s) = \frac{1}{6\sqrt{2}}\left(6s + s^3, 3\sqrt{2}s^2, 6s - s^3\right).$$

Corollary 10 *If the natural lift $\overline{\alpha}_B$ of α_B is an integral curve of the geodesic spray on the tangent bundle $T(\Lambda)$, then α_B is a geodesic on $\mathbb{R}^3_1$.*

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