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Bandwidth of Direct Products of Paths and Cycles

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Abstract

We give relatively simple proofs of both known and new results for the bandwidth of graphs involving direct products of paths and cycles. These include the rectangular lattice, the cylinder graph, the toroidal graph, and some more related results.

Keywords: Bandwidth of products of graphs, rectangular lattice, cylinder graph, toroidal graph

Introduction: Among all graph labelling problems, bandwidth numbering of graphs has perhaps attracted the most attention in the literature. The bandwidth numbering problem was proposed independently by Harper [9] and Harary [8]. Suppose that G is a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For undefined terminology we refer to [2]. A *labelling* f is a bijection $f : V \to X_n$ where $|V| = n$ and $X_n = \{1, 2, ..., n\}$. Let $F = \{f : V \to X_n, f \text{ a bijection}\}$. We define the *bandwidth of a labelling* f of G as $BW_f(G) = max_{uv\in E} |f(u) - f(v)|$. The *bandwidth of* G as in [4] is given by $BW(G) = min_{f\in F} \{max_{uv\in E} |f(u) - f(v)|\}$. We say that f is a *bandwidth labelling* of G if $BW_f(G) = BW(G)$. Let P_m, C_m denote respectively a path and a cycle on m vertices. It is known that the $BW(P_m \times P_n) = min\{m, n\}$

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[5, 7, 12], $BW(P_m \times C_n) = min\{2m, n\}$ [5] and the bandwidth of the toroidal graph $C_m \times C_n$ was shown to be $2m$ if $m < n$ and $2m - 1$ if $m = n$ [6, 13]. We consider all three types of graphs and give simple proofs that exploit a principle introduced by Bollobas [1; Problem 21].

Let G be a graph on $n = |V(G)|$ vertices or cells as we shall call them. Let $f : V(G) \to X_n = \{1, 2, ..., n\}$ be a labelling of the cells of the graph. We define *the colouring of* G *from* r as the colouring of all cells v with $f(v) \leq r$ as blue and the rest as red ($1 \leq r \leq n$). We denote a colour by c and the alternative colour by c'. We define a *c-coloured* cell to be a *star* if some adjacent cell is coloured c'. We now prove a lemma that is akin to Harper's Lemma [3; Theorem 3.2.2, 10], which is invoked in many verifications of bandwidths of particular classes of graph.

Lemma 1 (**Colouring Lemma**) If for every labelling f of the cells of G there is some colour c and some value of r (with c, r depending on f) such that G has at least k c-coloured stars, then $BW(G) \geq k$.

Proof Suppose the condition of the lemma is satisfied. If there are k blue stars then there is a blue star with label less than or equal to $r - k + 1$ adjacent to a red star labelled at least $r + 1$. The difference in labels of these adjacent cells is then at least $(r + 1) - (r - k + 1) = k$. Similarly if there are k red stars then there is a red star labelled at least $r + k$ adjacent to blue star labelled no more than r and the difference of their labels is at least k. It follows that the width of the labelling f is at least k for every labelling f and so $BW(G) \geq k$. □

We shall exploit the following inequality, the second part of which is to be found in [4].

Theorem 2 Let $G_i (i = 1, 2)$ be connected graphs, with $n_i = |V(G_i)| \geq 2$, $w_i = BW(G_i)$. Then for $G = G_1 \times G_2$

$$max\{min\{n_1, n_2), w_1, w_2\} \leq BW(G) \leq min\{n_1 w_2, n_2 w_1\}.$$

Proof Without loss we take $V(G_1) = \{1, 2, \cdots, n_1\}$ and $V(G_2) = \{1, 2, \cdots, n_2\}$. Define the *ith row* R_i and *jth column* C_j of G respectively as

$$R_i = \{(i, t), t = 1, 2, \cdots, n_2\}, C_j = \{(s, j), s = 1, 2, \cdots, n_1\}.$$

The general term that we shall use for either a row or a column is a *line*. Consider an arbitrary labelling f of G and let r denote the smallest maximum label that occurs in any line, which we shall first assume occurs in row $R = R_i$ and that r labels cell $c = (i, j)$. Consider the $(r-1)$-colouring. (Note that $r \geq 2$ as each of the G_i is non-trivial.) Observe that every line has a red cell and that c is the unique red cell in R. For $k \neq j$, the cell $u = (i, k)$ is coloured blue. Since G_2 is connected, there is a path in C_k from u to some cell in C_k that is coloured red. Hence C_k contains both a blue star and a red star. The same applies to C_j unless C_j is entirely red. However, since every cell in R apart from c is coloured blue and $n_2 \geq 2$, it follows that c is itself a red star in C_j as c is adjacent to some blue cell in R. Therefore there are at least n_2 red stars in G under the labelling f. Applying the same argument to the case where the cell c represents a column maximum we conclude that there are at least $min\{n_1, n_2\}$ red stars under the $(r-1)$-colouring, thus establishing that $min\{n_1, n_2\} \leq BW(G)$.

For the second part of the first inequality we note that the induced subgraph of G with vertex set $V(R_i)$ is isomorphic to G_2 and so we have $BW(G) \geq BW(R_i) = BW(G_2) = w_2$; similarly $BW(G) \geq w_1$ and these observations together with the preceding paragraph establish the first inequality.

For the second inequality we define a labelling f of G such that $|f(u) - f(v)| \leq n_1 w_2$ for all edges $uv \in E(G)$. The result then follows by symmetry. Label the members of each $V(G_i)$ in such a way that if $ij \in E(G_i)$ then $|i - j| \leq w_i$. Now $V(G) = \{(j_1, j_2) : 1 \leq j_i \leq n_i, i = 1, 2\}$. Label (j_1, j_2) by $(j_2 - 1)n_1 + j_1$, giving a bijection of $V(G)$ onto $X_{n_1 n_2}$. Suppose that $((j_1, j_2), (j_1, j_2')) \in E(G)$. Then the label difference is $|(j_2 - 1)n_1 + j_1 - (j_2' - 1)n_1 - j_1| = n_1 |j_2 - j_2'| \leq n_1 w_2$. Alternatively $((j_1, j_2), (j_1', j_2)) \in E(G)$ where the label difference is $|(j_2 - 1)n_1 + j_1 - (j_2 - 1)n_1 - j_1'| = |j_1 - j_1'| \leq w_1 < n_1 \leq n_1 w_2$. $\square$

Corollary 3 $BW(P_m \times P_n) = min\{m, n\}$.

Proof Since $BW(P_m) = 1$ all three terms in the inequalities in Theorem 2 equal $min\{m, n\}$, whence the result follows. $\square$

Corollary 4 $BW(P_m \times C_n) \leq min\{2m, n\}$ and $BW(C_m \times C_n) \leq 2\, min\{m, n\}$.

Proof Apply Theorem 2 to each case and use that $BW(P_m) = 1, BW(C_n) = 2$. $\square$

Lemma 5 Let G and H be non-trivial connected graphs with n_1 and n_2 vertices respectively and let $n_1 \leq n_2$. Then

$$max\{min\{n_1, n_2\},\, w_1, w_2\} = min\{n_1 w_2,\, n_2 w_1\}$$

if and only if $w_2 = 1$, in which case both sides equal n_1.

Remark In the case where $n_1 = n_2$, it may be necessary to interchange the subscripts 1 and 2 to make the claim true, as the following proof reveals.

Proof of Lemma 5 If $w_2 = 1$ then both sides equal n_1 (as $w_1 < n_1$) and we have equality.

Conversely suppose both expressions are equal. If the LHS $= w_1$ we get either $w_1 = n_2 w_1$, which implies that $n_2 = 1$, a contradiction, or $w_1 = n_1 w_2$, which is also impossible as $w_1 < n_1 \leq n_1 w_2$. If the LHS were w_2 we would get either that $w_2 = n_1 w_2$, which gives the contradiction that $n_1 = 1$ or $w_2 = n_2 w_1$, which contradicts the fact that $w_2 < n_2$. Hence the LHS is $min\{n_1, n_2\} = n_1$.

We now get that $n_1 = min\{n_1 w_2, n_2 w_1\}$. This implies either that $n_1 = n_1 w_2$, whence $w_2 = 1$ as required, or $n_1 = n_2 w_1$, whence, since $n_1 \leq n_2$, we get that $n_1 = n_2$ and $w_1 = 1$. In this latter case we can interchange the labels n_1 and n_2 to get the required conclusion that $w_2 = 1$ and that both sides equal n_1. $\square$

As a consequence we obtain the following:

Corollary 6 Let G be a connected graph on n vertices and P a path on at least n vertices. Then $BW(G \times P) = n$.

Proof Since $n \leq |V(P)|$ and $BW(P) = 1$, Lemma 5 applies and all terms in Theorem 2 equal n. Therefore $BW(G \times P) = n$. $\square$

We now give alternative proof of the next two propositions that arise in [11].

Proposition 7 Let K_m be the complete graph on m vertices then $BW(K_m \times P_n) = m$.

Proof $BW(K_m \times P_n) = m$ for $m \leq n$ by Corollary 6 , so assume that $m > n \geq 2$.

The inequality in Lemma 5 then becomes $max\{min\{m, n\}, m-1, 1\} \leq BW \leq min\{m, n(m-1)\}$ it follows that $max\{n, m-1, 1\} \leq BW \leq m$. Hence $m - 1 \leq BW \leq m$. However, since the minimum degree of the vertices in $K_m \times P_n$ is m, the former possibility is discounted as for any graph G, bandwidth is at least its minimum degree i.e. $BW(G) \geq \delta(G)$ and so the answer is indeed m in all cases. $\square$

Proposition 8 $BW(K_m \times C_n) = 2m$ for $m \geq 2, n \geq 4$.

Proof Let $G = K_m \times C_n$. We show that any subset S of $V(G)$ of order m has boundary ∂S of size at least $2m$ whence the result follows by Harper's

Lemma [9; Theorem 3.2] (as we know by Lemma 5 that $BW(G)$ is no more than $min\{2m, (m-1)n\} = 2m$).

Let S be a set of vertices of order m. We define the 'rows' as the subgraphs of G with a common first co-ordinate, which are then n-cycles and the 'columns' have a common second co-ordinate and are complete graphs on m vertices. If S is a column of G then $|\partial S| = 2m$. Suppose that S lies entirely in two columns of G with column C having k vertices of S and column D having $m-k$. Since $n \geq 4$ there exists a column E cyclically adjacent to C and a distinct fourth column F, cyclically adjacent to D. Then we see that $|\partial S|$ is at least $(m-k)+k+(m-k)+k = 2m$ (contributions from columns $C+E+F+D$ in that order).

Otherwise S has non-trivial intersections with t columns where $t \geq 3$. Let the size of these intersections be $k_1, k_2, \ \ldots \ k_t$ where each k_i is a positive integer. Then, because the columns are complete graphs, we have $|\partial S|$ is at least $(m-k_1)+(m-k_2)+ \ \ldots \ +(m-k_t) = tm-(k_1+k_2+ \ \ldots \ +k_t) = tm-m = (t-1)m$, which is at least $2m$ and the result follows. □

In the following proofs the strategy is to prove that $BW(G) \geq k$ by showing that the conditions of the Colouring Lemma are satisfied. The original proof of the next result [4] is a nine page calculation.

Theorem 9 $BW(P_m \times C_n) = k = min\{2m, n\}$.

Proof Let $G = (P_m \times C_n)$. By Corollary 4 we have $BW(G) \leq k$. We now prove the reverse inequality. Define r as the minimum of the line maxima as in Theorem 2, so the 'columns' are m copies of the cycles C_n. Consider the $(r-1)$-colouring, so that every row and cycle has a red cell. If there were fewer than three red rows, then every row of mixed colour would contain stars of both colours, and each red row R would contain a red star, as R is adjacent to a row containing a cell coloured blue. This would give n red stars. Otherwise there exist three red rows. Now consider the r-colouring. At least two of the three red rows remain red in this colouring. Since G can then have no blue cycle, the row containing the cell labelled r is a blue row. It now follows that every cycle contains two red stars, giving $2m$ red stars, and so the result follows by the Colouring Lemma. □

Our next candidate for bandwidth calculation is $G = C_m \times C_n$. Once again r denotes the minimum of the cycle maxima. We shall call a cycle a c-coloured *belt* if no more than one of its cells is coloured c'. A monochromatic c-belt will be called a *band*. The unique c'-coloured cell of a c-belt (if it has one) will be called its *buckle.* A cycle that is not a belt will be called an *ordinary* cycle. Throughout, a claim of the type that 'there are s objects…' will be understood to mean that there are at least s objects … .

Theorem 10 For $3 \leq m < n, BW(C_m \times C_n) = 2m$, while $BW(C_m \times C_m) = 2m - 1$.

Respective labellings of bandwidth $2m$ and $2m-1$ are constructed in [6]. We prove each of the reverse inequalities of this theorem in turn via a lemma, in an alternative fashion to [6].

Lemma 11 When $m \leq n$, $BW(C_m \times C_n) \geq 2m - 1$.

Proof Consider the r-colouring. By definition of r, there exists a blue band B whose orientation we label as *horizontal* with the set of cycles orthogonal to B being described as *vertical*. Observe that there are no other blue bands except perhaps the cycle B' orthogonal to B whose intersection with B is the cell labelled by r.

Suppose that in some direction (horizontal or vertical) there was no red belt. Every cycle in this direction, excepts perhaps B or B' has a red cell and two blue cells, and so two blue stars. Both B and B' are adjacent to a cycle with a red cell and so both B and B' have a blue star. This yields $2(m-1)+1 = 2m-1$ blue stars overall, as required. Moreover, if there were just one vertical red belt R', then R' would have one blue star and all other vertical cycles would have two blue stars, except perhaps B' for if B' were a blue band it might have only one blue star (which exists as the neighbouring vertical cycles have red cells). In this case, the $(r-1)$-colouring produces two blue stars in the cycle B', and in every vertical cycle except R', which has one, giving $2m-1$ blue stars in all. Hence we continue under the assumption that in the r-colouring there exists at least two vertical red belts R_1' and R_2', at least one horizontal red belt R_1, and a horizontal blue band B.

Suppose now that there exists no more than one vertical blue belt. Then there exists $m-1$ vertical cycles with two red cells, which each meets the blue band B. This yields $2(m-1) = 2m-2$ red stars. If the remaining vertical cycle is not a blue band, it too has a red star, yielding the required number of $2m-1$ red stars. Hence we may assume that the remaining vertical cycle is a blue band, which is necessarily B'. Consider again the $(r-1)$-colouring. Then there are two red stars in each vertical cycle except B', which now has one, giving $2m-1$ red stars.

Hence we may assume that there exists two vertical blue belts, B_1', B_2', two vertical red belts R_1' and R_2', and there exists a horizontal blue band B (and a horizontal red belt R). By re-subscripting the cycles as necessary we may assume that in some cyclic direction, we may travel orthogonally from R_1' to B_1' without first meeting the other named vertical belts, and similarly we may travel in some (perhaps opposite) cyclic direction from R_2' to B_2' without meeting the other two vertical belts. Take any red cell in R_i' $(i = 1, 2)$ and travel in the direction of B_i'; this will lead to a red star before we meet B_i', with perhaps one exception that will lie in B_i' itself and the blue neighbour (indeed two of them) will be vertically instead of horizontally adjacent. This gives $2(m-1) = 2m-2$ red stars overall. Again we consider the $(r-1)$-colouring, in which the cell r becomes another red star. This does give a set of $2m-1$ red stars overall, although the red star identified in moving from the cell c in R_i' that is adjacent to B may no longer be a red star, there remains a red star in the sequence from c towards the blue belt B_i'. This completes the proof of the claim that the bandwidth is bounded below by $2m-1$.

Corollary 12 $BW(C_m \times C_m) = 2m-1$.

Proof By [6] we have $BW(C_m \times C_m) \leq 2m-1$ and by Lemma 11 we have $BW(C_m \times C_m) \geq 2m-1$, whence the result follows. $\square$

Lemma 13 For $m < n$, $BW(C_m \times C_n) \geq 2m$.

Proof We proceed using the ideas and notation of the previous lemma. Consider the $(r-1)$-colouring, so that every cycle has a red cell.

Case 1 First assume that there are at least $m+1$ vertical cycles (i.e. $m+1$ cycles orthogonal to B). We shall suppose to the contrary that there are fewer than $2m$ stars of each colour. Each ordinary vertical cycle contains two stars of each colour. If there were fewer than two vertical blue belts then, since B is a horizontal blue belt, there would exist $m-1$ vertical cycles with two red stars, giving $2m-2$ red stars. There cannot be two vertical red bands, because of the presence of B so one of the remaining vertical cycles is not a red band and so has one red star. Any vertical red band also has one red star as it is adjacent to a cycle that has a blue cell. Hence there would be at least $2m$ red stars. Therefore we may assume the existence of two vertical blue belts B_1 and B_2, which are not bands.

If there were fewer than two vertical red belts then we would have m vertical cycles with two blue stars giving $2m$ blue stars and so we may assume that there are two vertical red belts as well as two vertical blue belts.

Suppose that there were exactly two vertical red belts. Then the other vertical cycles contain $2m-2$ blue stars and each vertical red belt contributes another blue star unless one is a band. If that is the case then the red band meets the horizontal blue belt in the cell labelled r. Then in this case the r-colouring produces a blue star in what was formerly a vertical red band without reducing the stated number of blue stars in the other vertical cycles, thus, in either event, we have $2m$ blue stars overall in either the $(r-1)$-colouring or the r-colouring. Hence we may assume there are three vertical red belts and two vertical blue belts in the $(r-1)$-colouring.

We can now write down two blue-red pairs of vertical belts, (B_1, R_1) and (B_2, R_2). By the argument of Lemma 11, there are $2m-2$ red stars identifiable in moving between the R_i and the B_i. By the same token, there are $2m-2$ blue stars. If there were now a third vertical blue belt, it would contain two additional blue stars, giving $2m$ blue stars overall. Hence we may assume that there are exactly two vertical blue belts.

Now a vertical red belt will have two red stars except if it is a red band flanked by two red belts with their blue buckles in the same horizontal cycle. We may assume that this applies to the third red belt R for otherwise we would have $(2m-2)+2 = 2m$ red stars. Hence we may now assume that there are exactly two vertical blue belts and one vertical red band R that is flanked by two red belts R_1, R_2.

As above, R has one red star, giving $2m-1$ red stars. Where R meets the blue belt B is then the cell labelled r. Now in this case the r-colouring increases the number of red stars in R to 2 without changing the red stars in the other vertical cycles, giving the required $2m$ red stars. This completes the proof in this case.

Case 2 The final set of circumstances is where there are $m+1$ horizontal cycles (and m vertical ones). Consider the r-colouring. If there were a vertical blue band, we would return to the previous case, so we assume otherwise. If there were no vertical blue (resp. red) belt, then every vertical cycle would have two red (resp. blue) stars, giving $2m$ red (resp. blue) stars, and so we may assume the existence of both kind of coloured vertical belt. It follows from this that there is no more than one horizontal red band.

We next consider the $(r-1)$-colouring. We may no longer assume the existence of a vertical blue belt but the inference of no more than one horizontal red band stands, as r is the label of the buckle of the blue belt B. Suppose that there was only one horizontal blue belt (B). Then there would be $m-1$ horizontal cycles with two red stars; also B has a red star, as does the horizontal red band, if it exists, giving $2m$ red stars. Hence we assume that there are two horizontal blue belts, $B_1 = B$ and B_2. If there were no more than one horizontal red belt, then there would be m horizontal cycles with two blue stars, giving $2m$ blue stars and so we may assume that there are two horizontal red belts, R_1 and R_2. By the argument of the previous case, there are $2(m-1) = 2m-2$ red stars and $2m-2$ blue stars identifiable in moving between the R_i and the B_i.

If there were three horizontal blue belts, this would yield $2m$ blue stars, so we may assume there are exactly two. If there were only two horizontal red belts, even if one were a red band, each would have two red stars and so there would be $2(m-1)+2 = 2m$ red stars, as each of the B_i has a red star. Hence there is a third horizontal red belt, which will yield another two red stars in addition to the $2m-2$ identified in the previous paragraph, giving $2m$ in all if there is no horizontal red band. Hence we may assume that there is a horizontal red band R. Moreover we may assume that R is flanked by R_1 and R_2, which are red belts with their buckles in the same vertical cycle. This gives $2m-1$ red stars.

We now note that if the number of horizontal cycles were at least $m+2$, there would be $2m+1$ red stars, so we may assume that there are exactly $m+1$ horizontal cycles. For there to be no more than $2m-1$ red stars, no horizontal cycle may contain more than two red stars. However, between some blue belt B_1 and a red belt R_1 there can be no more than $(m+1-5)/2 = (m-4)/2$ ordinary cycles. In order to avoid more than two red stars occurring in some cycle, the number of cells of each colour in two adjacent cycles must differ by no more than two. If the difference were exactly two on each occasion the increase in the number of red cells per cycle in travelling from B_1 to R_1 would be bounded above by $2 \times \frac{m-4}{2} + 2 = m-2$, which is exactly the increase required (and the number of red cells in each ordinary cycle would be odd). However it follows from this that there must be exactly $\frac{m-4}{2}$ ordinary horizontal cycles separating B_i from R_i ($i = 1, 2$) (and that the two B_i are mutually adjacent with m even). However, for each ordinary cycle to have exactly two red stars, its sets of blue and red cells must be contiguous. Let C and D be two adjacent horizontal cycles, with C having exactly two fewer red cells than D, which lie between B_i and R_i, including perhaps the case where $C = B_i$ or $D = R_i$. The block of red cells in C must be exactly adjacent to a block of red cells of the same length in D, with the red block in D further extended by one red cell in each direction. However, in that case every star in an ordinary cycle now has *two* adjacent cells of the opposite colour, one in its horizontal cycle and one in a flanking cycle and this is also true of belt stars, except that the buckle of each belt has two neighbours of the opposite colour in the belt itself. It now follows that there is a red star with label at least $r + 2m - 1$ that has a blue neighbour with label no more than $r - 1$. This gives a label difference of two adjacent cells of at least $(2m + r - 1) - (r - 1) = 2m$, as required to complete the proof. $\square$

Corollary 14 For $m < n$, $BW(C_m \times C_n) = 2m$.

Proof By Corollary 4 we have $BW(C_m \times C_n) \leq 2m$, and by Lemma 13 $BW(C_m \times C_n) \geq 2m$, whence the result follows $\square$

Proof of Theorem 10 Corollaries 12 and 14 together give the required result. $\square$

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