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On Matrix Near Ring

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Abstract. Let R be a right near ring with identity and $\mathrm{M}_n(R)$ be the near ring of $n \times n$ matrices over R . Let R_0, R_c ,R_d $((\mathrm{M}_n(R))_0, (M_n(R))_c, (\mathrm{M}_n(R))_d)$ be the zero symmetric, constant and distributive parts of R ($\mathrm{M}_n(R)$) respectively. In this paper we show that $(\mathrm{M}_n(R))_0 \cong \mathrm{M}_n(R_0)$, $(\mathrm{M}_n(R))_c \cong \mathrm{M}_n(R_c)$ and $(\mathrm{M}_n(R))_d \cong \mathrm{M}_n(R_0)$. Also we define the class of generalized abstract affine near rings that contains the class of abstract affine near ring and if R belongs to this class then $\mathrm{M}_n(R) \cong \mathrm{M}_n(R_0) + \mathrm{M}_n(R_c)$.

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1. Introduction

Let $R = R_0 + R_c$ is a right near ring, $\mathrm{M}_n(R)$ be the near ring of $n \times n$ matrices over R in the sence of melderum and Van der Walt. Let $R_0 = \{r \in R : r0 = 0\}, R_c = \{r \in R : rx = r \forall x \in R\}$ and $R_d = \{r \in R : r(x + y) = rx + ry \forall x, y \in R\}$ are the zero symmetric, constant and distributive parts of R., $\mathrm{M}_n(R))_0, (M_n(R))_c$ and $(\mathrm{M}_n(R))_d$ are the zero symmetric, constant and distributive parts of $\mathrm{M}_n(R)$. Also let $\mathrm{M}_n(R_0)$,$\mathrm{M}_n(R_c)$ and $\mathrm{M}_n(R_d)$ be the near ring of $n \times n$ matrices over R_0, R_c and R_d respectively. In this paper we show that

(i) the zero symmetric part of $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_n(R_0)$ i.e $(\mathrm{M}_n(R))_0 \cong \mathrm{M}_n(R_0)$ as in lemma2.1

(ii) the constant part of $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_n(R_c)$ i.e $(\mathrm{M}_n(R))_c \cong \mathrm{M}_n(R_c)$ as in lemma2.2

(iii) the distributive part of $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_n(R_d)$ i.e $(\mathrm{M}_n(R))_d \cong \mathrm{M}_n(R_d)$ as in lemma2.3.

A right near ring $R = R_0 + R_c$ is said to be an abstract affine near ring if

(a) R is abelian near ring

(b) $R_0 = R_d$.

Let A be a ring and M a left module over A . By [2 ,Prop.4] there is exactly one way to extend the multiplication $\cdot : A \times M \rightarrow M$ to a multiplication "$\circ$" in $(R,+) = (A,+) \oplus (M,+)$ such that $(R,+,\circ)$ is a right near-ring with $R_d = R_0 = A \oplus (0)$ and $R_c = (0) \oplus M$, namely $(r,m) \circ (s,n) = (r \cdot s, r \cdot n + m)$. $R = (R,+,\circ)$ is an abstract affine near-ring, this abstract affine near-ring will be denoted by $A * M$. Moreover all abstract affine near-rings arise in this way. Conversely, for any abstract affine near-ring R,the zero symmetric part R_0 is a ring and the constant part $R_c = \{a \in R : a.x = a$,for all $x \in R\}$ is a left module over R_0.
In this paper we define the class of generalized abstract affine right near rings as in definition2.4 which contains the class of abstract affine right near rings. Also we show that if $R = R_0 + R_c$ is a generalized abstract affine near ring then $\mathrm{M}_n(R) \cong \mathrm{M}_n(R_0) + \mathrm{M}_n(R_c)$ as in Theorem 2.6.

2. Main Results

The following lemma shows that the zero symmetric part of $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_n(R_0)$
Lemma 2.1
If $R = R_0 + R_c$ is a right near ring then

$$(\mathrm{M}_n(R))_0 \cong \mathrm{M}_n(R_0)$$

Proof
We show that $\mathrm{M}_n(R_0)$ is embeded in $(\mathrm{M}_n(R))_0$ and there exist a one to one correspondence betwean the set of generators of $(\mathrm{M}_n(R))_0$ and the set of generators of $\mathrm{M}_n(R_0)$. First we show that $\mathrm{M}_n(R_0)$ is embeded in $(\mathrm{M}_n(R))_0$. Define

$$\begin{aligned} h_0 \quad &: \quad \mathrm{M}_n(R_0) \rightarrow \mathrm{M}_n(R) \\ A \quad &\rightarrow \quad h_0(A) \text{ where } h_0(A) : R^n \rightarrow R^n \text{ such that } h_0(A)|_{R_0^n} = A \text{ and } h_0(A)|_{R^n - R_0^n} = O \end{aligned}$$

where $h_0(A)|_{R_0^n}$ is the restriction of $h_0(A)$ to R_0^n and $h_0(A)|_{R^n - R_0^n}$ is the restriction of $h_0(A)$ to $R^n - R_0^n$.It is clearly that $h_0(A) \in \mathrm{M}_n(R)$ we show that
[a]Φ is a homorophism
let $A, C \in \mathrm{M}_n(R_0)$ then it is clearly that

$$h_0(A + C) = h_0(A) + h_0(C) \quad (i)$$

$$h_0(AC) = h_0(A)h_0(C) \quad (ii)$$

(ii) follows since

$$h_0(AC)|_{R^n - R_0^n} = O = (h_0(A)h_0(C))|_{R^n - R_0^n}$$

and

$$h_0(AC)|_{R_0^n} = AC = (h_0(A)h_0(C))|_{R_0^n}$$

so Φ is a homorophism

[b] Φ is injective since if

$$\Phi(A) = \Phi(C)$$

then

$$h_0(A) = h_0(C)$$

and so

$$A = h_0(A)|_{R_0^n} = h_0(C)|_{R_0^n} = C$$

then from [a],[b] we have $\mathrm{M}_n(R_0)$ is embeded in $\mathrm{M}_n(R)$. But $\mathrm{M}_n(R_0)$ is zero-symmetric so $\mathrm{M}_n(R_0)$ is embeded in $(\mathrm{M}_n(R))_0$ i.e

$$\mathrm{M}_n(R_0) \hookrightarrow (\mathrm{M}_n(R))_0.$$

Second we show that there exist a one to one correspondence betwean the set of generators of $(\mathrm{M}_n(R))_0$ and the set of generators of $\mathrm{M}_n(R_0)$. It is clearly that

$$f_{ij}^r \in (\mathrm{M}_n(R))_0 \text{ iff } r \in R_0$$

becouse

$$f_{ij}^r(0, 0, ..., 0) = (0, 0, ..., 0) \text{ iff } (0, ..., r0, ..., 0) = (0, 0, ..., 0)$$

$$\text{iff } r0 = 0 \text{ iff } r \in R_0$$

so there exist a one to one correspondence between the generators of $\mathrm{M}_n(R_0)$ and the generators of $(\mathrm{M}_n(R))_0$ and then we have

$$\mathrm{M}_n(R_0) \cong (\mathrm{M}_n(R))_0 \quad .\blacksquare$$

The following lemma shows that the constant part of $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_n(R_c)$

Lemma2.2.
if $R = R_0 + R_c$ is a right near ring then

$$(\mathrm{M}_n(R))_c \cong \mathrm{M}_n(R_c)$$

Proof
We show that $\mathrm{M}_n(R_c)$ is embeded in $(\mathrm{M}_n(R))_c$ and there exist a one to one correspondence betwean the set of generators of $(\mathrm{M}_n(R))_c$ and the set of generators of $\mathrm{M}_n(R_0)$. First we show that $\mathrm{M}_n(R_c)$ is embeded in $(\mathrm{M}_n(R))_c$.Define

$$\begin{aligned} h_c &: \mathrm{M}_n(R_c) \to \mathrm{M}_n(R) \\ A &\to h_c(A) \text{where } A(a_1, ..., a_n) = (b_1, ..., b_n)\ \forall (a_1, ..., a_n) \in R_c^n \text{ we define} \\ h_c(A) &: R^n \to R^n \text{ such that } h_c(A)(x_1, ..., x_n) = (b_1, ..., b_n) \forall (x_1, ..., x_n) \in R^n \end{aligned}$$

it is clearlly that $\mathrm{M}_n(R_c)$ is a subnear ring of $\mathrm{M}_c(R_c^n)$ we have
[a] Φ is a homorophism since, let $A, C \in \mathrm{M}_n(R_c)$ then

$$\begin{aligned} h_c(A+B)(a_1,...,a_n) &= (A+B)(a_1,...,a_n) \\ &= A(a_1,...,a_n)+B(a_1,...,a_n) \\ &= h_c(A)(a_1,...,a_n)+h_c(B)(a_1,...,a_n) \\ &= (h_c(A)+h_c(B))(a_1,...,a_n) \end{aligned}$$

$$\begin{aligned} h_c(AB) &= h_c(A) \\ &= h_c(A)h_c(B) \quad (ii) \end{aligned}$$

(ii) follows since $\mathrm{M}_n(R_c)$ is a subnear ring of $\mathrm{M}_c(R_c^n)$.
[b] Φ is injective since if

$$h_c(A) = h_c(C)$$

so we have

$$A = C$$

Second we show that there exist a one to one correspondence betwean the set of generators of $(\mathrm{M}_n(R))_c$ and the set of generators of $\mathrm{M}_n(R_c)$.It is clearly that

$$f_{ij}^r \in (\mathrm{M}_n(R))_c \text{ iff } r \in R_c$$

becouse

$$\begin{aligned} f_{ij}^r O &= f_{ij}^r \text{ iff } (0,...,r0,...,0) = (0,0,...,r,...,0) \\ \text{iff } r0 &= r \text{ iff } r \in R_c \end{aligned}$$

so there exist a one to one correspondence betwean the generators of $\mathrm{M}_n(R_c)$ and $(\mathrm{M}_n(R))_c$ and then we have

$$\mathrm{M}_n(R_c) \cong (\mathrm{M}_n(R))_c. \blacksquare$$

The following lemma shows that the distributive part of $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{M}_n(R_d)$

Lemma 2.3
if $R = R_0 + R_c$ is a right near ring then

$$(\mathrm{M}_n(R))_d \cong \mathrm{M}_n(R_d)$$

Proof
Proof
We show that $\mathrm{M}_n(R_d)$ is embeded in $(\mathrm{M}_n(R))_d$ and there exist a one to one correspondence betwean the set of generators of $(\mathrm{M}_n(R))_d$ and the set of generators of $\mathrm{M}_n(R_d)$. First we show that $\mathrm{M}_n(R_d)$ is embeded in $(\mathrm{M}_n(R))_d$. Since R_d is embeded in R then $\mathrm{M}_n(R_d)$ is embeded in $\mathrm{M}_n(R)$ i.e $\mathrm{M}_n(R_d) \hookrightarrow \mathrm{M}_n(R)$. But $\mathrm{M}_n(R_d)$ is distributive so $\mathrm{M}_n(R_d)$ is embeded in $(\mathrm{M}_n(R))_d$ i.e

$$\mathrm{M}_n(R_d) \hookrightarrow (\mathrm{M}_n(R))_d.$$

Second we show that there exist a one to one correspondence betwean the set of generators of $(\mathrm{M}_n(R))_d$ and the set of generators of $\mathrm{M}_n(R_d)$.It is clearly that

$$f_{ij}^r \in (\mathrm{M}_n(R))_d \text{ iff } r \in R_d$$

so there exist a one to one correspondence betwean the generators of $\mathrm{M}_n(R_d)$ and $(\mathrm{M}_n(R))_d$ and then we have

$$\mathrm{M}_n(R_d) \cong (\mathrm{M}_n(R))_d.\blacksquare$$

Definition2.4
A right near ring $R = R_0 + R_c$ is said to be a generalized abstract affine near ring if
(1) $n_0 + n_c = n_c + n_0 \quad \forall n_0 \in R_0, \forall n_c \in R_c$
(2) $R_0 = R_d$.

It is clearly that every abstract affine near ring is a generalized abstract affine near ring.
The following example is an example of a generalized abstract affine near ring which is not abstract affine near ring
Example 2.5
Let R be any ring, M be non abelian group such that there is a map

$$\begin{array}{rcl} . & : & R \times M \to M \\ (r, m) & \to & r.m \end{array}$$

that satisfies the following conditions
(i) $(rs).m = r.(s.m) \forall r, s \in R, \forall m \in M$
(ii) $(r + s).m = r.m + s.m \forall r, s \in R, \forall m \in M$
(iii) $r.(m + n) = r.m + r.n \forall r \in R, \forall m, n \in M$

Take $S = R \oplus M$ be the direct sum of the two groups R, M. Under the component with addition and the following multiplication $o : S \times S \to S$ defined by $(r, m)o(s, n) = (rs, rn)$, S is a right near ring. It is clearlly that S is not abstract affine near ring becouase it is non abelian. On the other hand S is generalized abstract affine near ring, where the condition (1) in definition 2.4 is satisfied since $S = R \oplus M$ be the direct sum of the two groups R, M, also condition (2) in definition 2.4 is satisfied.$\blacksquare$

Theorem 2.6
If $R = R_0 + R_c$ is a generalized abstract affine near ring then

$$\mathrm{M}_n(R) \cong \mathrm{M}_n(R_0) + \mathrm{M}_n(R_c)$$

Proof
We have

$$\mathrm{M}_n(R) = (\mathrm{M}_n(R))_0 + (\mathrm{M}_n(R))_c$$

so from lemma2.1,lemma 2.2 we have

$$\begin{array}{rcl} h_0 & : & \mathrm{M}_n(R_0) \to (\mathrm{M}_n(R))_0 \\ A & \to & h_0(A) \text{ where } h_0(A) : R^n \to R^n \text{ such that } h_0(A)|_{R_0^n} = A \text{ and } h_0(A)|_{R^n - R_0^n} = O \end{array}$$

is an isomorphism and

$$\begin{array}{rcl} h_c & : & \mathrm{M}_n(R_c) \to (\mathrm{M}_n(R))_c \\ A & \to & h_c(A) \text{where } A(a_1, ..., a_n) = (b_1, ..., b_n)\ \forall (a_1, ..., a_n) \in R_c^n \text{ we define} \\ h_c(A) & : & R^n \to R^n \text{ such that } h_c(A)(x_1, ..., x_n) = (b_1, ..., b_n) \forall (x_1, ..., x_n) \in R^n \end{array}$$

is also an isomorphism. But it is clearly from [2] that

$$\begin{array}{rcl} h & : & S = \mathrm{M}_n(R_0) + \mathrm{M}_n(R_c) \to (\mathrm{M}_n(R))_0 + (\mathrm{M}_n(R))_c = \mathrm{M}_n(R) \\ h(A + B) & = & h_0(A) + h_c(B) \end{array}$$

is a homorophism if and only if the following conditions holds
(a) $h_0(A) + h_c(B) = h_c(B) + h_0(A)$
(b) $h(AC) = h_0(A)h(C)\ \forall A \in \mathrm{M}_n(R_0), \forall C \in S$.
[1]The condition (a) obtains from the condition (1) of definition 2.4 and the definitions of h_0, h_c. To satisfies the condition (b) we have if $C = D + L, D \in S_0 = S_d, L \in S_c$ then we have from the the condition (2) of definition 2.4and lemma2.3 that $\mathrm{Mat}_n(R_0) \cong (\mathrm{M}_n(R))_0$ is distributive so we have

$$\begin{array}{rcl} h(AC) & = & h(A(D + L)) \\ & = & h(AD + AL) \\ & = & h_0(AD) + h_c(AL) \quad (i) \end{array}$$

where $AD \in S_0, AL \in S_c$.Also we have

$$\begin{array}{rcl} h_0(A)h(C) & = & h_0(A)h(D + L) \\ & = & h_0(A)(h_0(D) + h_c(L)) \\ & = & h_0(A)h_0(D) + h_0(A)h_c(L) \quad \text{since } S_0 = S_d \quad (ii) \end{array}$$

so $h(AC) = h_0(A)h(C)$ iff $h_c(AL) = h_0(A)h_c(L)$ but $h_c(AL) = h_0(A)h_c(L) = 0$ in this case so we have h is a homorophism.
[2]h is one to one becouse if

$$\begin{array}{rcl} h(A) & = & h(B) \\ h_0(A_0) + h_c(A_c) & = & h_0(B_0) + h_c(B_c) \end{array}$$

so

$$h_0(A_0) - h_0(B_0) = h_c(B_c) - h_c(A_c)$$

but $h_0(A_0) - h_0(B_0) \in S_0$, $h_c(B_c) - h_c(A_c) \in S_c$ and $S_0 \cap S_c = \{0\}$ then

$$h_0(A_0) - h_0(B_0) = h_c(B_c) - h_c(A_c) = 0$$

and so we have

$$h_0(A_0) = h_0(B_0), h_c(B_c) = h_c(A_c)$$

but h_0, h_c are one to one then

$$A_0 = B_0, A_c = B_c$$

so $A = B$.
[3] Finally it is clearly that h is onto since h_0, h_c are onto . So from [1],[2] and [3] h is an isomorphism.■

References

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