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Generalized Matrix Near Ring over Abstract Affine Near Ring

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Abstract. Let A be an abstract affine near ring, M be a faithful near ring $A-$module and n be a positive integer. In this paper we define the $n \times n$ generalized matrixe near ring over A using the faithful near ring $A-$modul M which is denoted by $\mathrm{Mat}_n(A, M)$. Also we find a necessary and sufficent condition for which $\mathrm{Mat}_n(A, M)$ is an abstract affine near ring.

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1. Introduction

The theory of near-rings is presented in[3].We recall some concepts of this theory.Let $A = (A, +, \cdot)$ be an abstract affine near-ring (a.a.n.r for short), i.e.$(A, +)$ is an abelian group,$(A, \cdot)$ is a semigroup, $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in A$, and $A_0 = A_d$, where $A_0 = \{a \in A : a \cdot 0 = 0\}$ is the zero symmetric part and A_d =$\{a \in A : a \cdot (x+y) = a \cdot x + a \cdot y$,for all $x, y \in A\}$ is the distributive part. Let R be a ring and M a left $R-$module.By [3, Prop.9.81] there is exactly one way to extend the multiplication $\cdot : R \times M \to M$ to a multiplication "$\circ$" in $(A, +) = (R, +) \oplus (M, +)$such that $(A, +, \circ)$ is a near-ring with $A_d = A_0 = R \oplus (0)$ and $A_c = (0) \oplus M$,namely $(r, m) \circ (s, n) = (r \cdot s, r \cdot n + m)$.Moreover ,$A = (A, +, \circ)$ is an aanr and all aanr's arise in this way.This aanr will be denoted by $R * M$.
Conversely , for any aanr A,the zero symmetric part A_0 is a ring and the constant part $A_c = \{a \in A : a.x = a$,for all $x \in A\}$ is a left module over A_0.
Let A be an abstract affine near ring, N be a faithful near ring $A-$module and n be a positive integer. In this paper we define the $n \times n$ generalized matrixe near ring over A using the faithful near ring $A-$modul N which is denoted by $\mathrm{Mat}_n(A, M)$ as the direct sum of the two near rings $\mathrm{Mat}_n(A_0, N)$

and A_c^n, where $\mathrm{Mat}_n(A_0, N)$ is the generalized matrix near ring as in [2], A_c^n is the direct sum of n copies of A_c which is a near ring under component wise addition and multiplication.
It is clearly that the direct sum of two right near rings S, D which is denoted by $S \oplus D$ is a right near ring under component wise addition and multiplication.
In the following $\oplus$ means direct sum, this lemma is important in our work

Lemma 1.1

Let S_1 and D_1 are right zero symmetric near rings and they are isomorphic (i.e $S_1 \cong D_1$), S_2 and D_2 are right constant near rings and isomorphic (i.e $S_2 \cong D_2$), then $S = S_1 \oplus S_2, D = D_1 \oplus D_2$ are isomorphic (i.e $S \cong D$).

Proof

Let $f_1 : S_1 \to D_1$, $f_2 : S_2 \to D_2$ are isomorphisms. Define $f : S \to D$, by $f(s_1, s_2) = (f_1(s_1), f_2(s_2))$.
It is clearly that f is an isomorphism.■
From [3,] if $A = R * M$,be an abstract affine near-ring and $n > 1$be a natural number.Then $\mathrm{M}_n(R * M) \simeq M_n(R) * {}^nM$.
In this paper we extend this to define the generalized matrix near ring $\mathrm{M}_n(A, M)$ when A is an abstract affine near ring.

2. Notations and Definitions

If A is a right near ring with identity, n be any positive integer. In 1986 J.D.P.Meldrum and A.P.J. Van der walt define Matrix near ring over A ,$\mathrm{Mat}_n(A)$, regards A as a left module over A.In [2] A denote a right, zero symmetric near ring with identity, M be a faithful left $A-$module, $(M, +)$ need not be a belian group and M^n is the direct sum of n copies of M, also a faihful left $A-$module.
Kirby.C.Smith in [2] define the $n \times n$ generalized matrix near ring over A using the faihful left $A-$module M as the subnear ring $\mathrm{M}_n(A, M)$ of $\mathrm{M}_0(M^n)$ generated by f_{ij}^r ,$r \in R$ and $1 \le i, j \le n$, where the generalized $n \times n$ matrix near ring will be a function from M^n to M^n. Now we define special functions in $\mathrm{M}_0(M^n)$ will be denoted by f_{ij}^r ,$r \in A$ and $1 \le i, j \le n$

$$\begin{aligned} f_{ij}^r &: M^n \to M^n \text{ such that } r \in A \text{ and } 1 \le i, j \le n, \\ f_{ij}^r(a_1, ..., a_k) &= (0, ..., 0, ra_j, 0, ..., 0) \text{ where } ra_j \text{ in the } i-\text{th position}, \\ (a_1, ..., a_k) &\in M^n \text{and } f_{ij}^r = l_i f^r \pi_j \text{ where } l_i : M \to M^n \text{is} \\ \text{the} i-\text{th injection}, \pi_j &: M^n \to M \text{ is the } j-\text{th projection and } f^r : M \to M \\ \text{such that } f^r(s) &= rs \ \forall s \in M , \end{aligned}$$

So f_{ij}^r is the function from M^n to M^n that takes a $n-$tuble with entries from M, multiples the $j-$th entry a_j by r using the module action of R on M, puts the result ra_j into the $i-$th position and puts 0 in the other positions. We may sometimes write f_{ij}^r as $[r; i, j]$.

3. Main Results

Definition 3.1
Let A be any abstract affine near ring, N be a faithful $A-$module then generalized matrix near ring with respect to A and N which is denoted by $\mathrm{Mat}_n(A,N)$ is the direct sum of $\mathrm{Mat}_n(A_0,N)$ and A_c^n i.e

$$\mathrm{M}_n(A,N) = \mathrm{M}_n(A_0,N) \oplus A_c^n$$

In the following $A = A_0 * A_c$ be any abstract affine near ring, N, M are faithful $A-$modules, n be a positive integer

Lemma 3.2

$$\mathrm{M}_1(A,N) \cong A \cong \mathrm{M}_1(A)$$

Proof
We have

$$\mathrm{M}_1(A,N) = \mathrm{M}_1(A_0,N) \oplus A_c$$

but also we have

$$\mathrm{M}_1(A_0,N) \cong \mathrm{M}_1(A_0) \cong A_0$$

and

$$A_c \cong \mathrm{M}_1(A_c).$$

So by lemma 1.1

$$\begin{aligned}\mathrm{M}_1(A,N) &= \mathrm{M}_1(A_0,N) \oplus A_c \cong A_0 \oplus A_c = A \\ &\cong \mathrm{M}_1(A_0) \oplus \mathrm{M}_1(A_c) = \mathrm{M}_1(A).\end{aligned}$$

Theorem 3.3
If $\theta : M \to N$ is an $A-$epimorphism, then θ induces a near ring epimorphism from $\mathrm{M}_n(A;M)$ into $\mathrm{M}_n(A;N)$. So if N is a homomorphic image of M, then the matrix near ring $\mathrm{M}_n(A;N)$ is a homomorphic image of $\mathrm{M}_n(A;M)$.
Proof
Since M, N are faithful $A-$modules so M, N are faithful A_0-modules and so we have

$$\mathrm{M}_n(A,M) = \mathrm{M}_n(A_0,M) \oplus A_c^n$$

$$\mathrm{M}_n(A,N) = \mathrm{M}_n(A_0,N) \oplus A_c^n.$$

But from [2, Theorem1] we have $\mathrm{M}_n(A_0;N)$ is a homomorphic image of $\mathrm{M}_n(A_0;M)$.So $\mathrm{M}_n(A,M) = \mathrm{M}_n(A_0,M) \oplus A_c^n$ is a homomorphic image of $\mathrm{M}_n(A,N) = \mathrm{M}_n(A_0,N) \oplus A_c^n$.
Corollary 3.4
Let M be a homomorphic image of ${}_AA$. Then $\mathrm{M}_n(A;M)$ is isomorphic to $\mathrm{M}_n(A;A)$ for all $n \geq 1$.
Proof
Since M is a faithful$A-$module which is a homomorphic image of ${}_AA$ so M

be a faithfulA_0-module which is a homomorphic image of ${}_{A_0}A_0$. Then from [2,Corollary1]

$$\mathrm{M}_n(A_0, M) \cong \mathrm{M}_n(A_0, A_0)$$

and so

$$\begin{aligned} \mathrm{M}_n(A; M) &= \mathrm{M}_n(A_0, M) \oplus A_c^n \\ &\cong \mathrm{M}_n(A_0, A_0) \oplus A_c^n \ \text{ from lemma1.1} \\ &\cong \mathrm{M}_n(A_0) \oplus A_c^n \quad \text{from lemma1.1} \\ &\cong \mathrm{M}_n(A) \text{ from [3,Lemma 2.1]} \\ &= \mathrm{M}_n(A; A) \end{aligned}$$

Corollary 3.5
If M and N are isomorphic faithful$A-$modules, then

$$\mathrm{M}_n(A; M) \simeq \mathrm{M}_n(A; N) \text{ for all } \ n \geq 1.$$

Proof
Since M and N are isomorphic faithful$A-$modules, then M and N are isomorphic faithfulA_0-modules and so from [2,Corollary2]

$$\mathrm{M}_n(A_0; M) \simeq \mathrm{M}_n(A_0; N) \text{ for all } \ n \geq 1.$$

Now for all $n \geq 1$

$$\begin{aligned} \mathrm{M}_n(A; M) &= \mathrm{M}_n(A_0; M) \oplus A_c^n \\ &\cong \mathrm{M}_n(A_0; N) \oplus A_c^n \text{ from lemma1.1} \\ &= \mathrm{M}_n(A_0; N)\ . \end{aligned}$$

Lemma 3.6

$$\mathrm{M}_n(A_0, A) \cong \mathrm{M}_n(A_0, A_0)$$

Proof
From [2, Corollary1] since A is a modul over A_0 under the usual multiplication and is a homorophic image of the module A_0 over A_0 ■
Proposition 3.7

$$\mathrm{M}_n(A, A) \cong \mathrm{M}_n(A)$$

Proof

$$\begin{aligned} \mathrm{M}_n(A, A) &= \mathrm{M}_n(A_0, A) \oplus A_c^n \text{ from definition 3.1} \\ &\cong \mathrm{M}_n(A_0, A_0) \oplus A_c^n \text{ from lemma 2.6, lemma1.1} \\ &\cong \mathrm{M}_n(A_0) \oplus A_c^n \text{ from [2, proposition1] and lemma1.1} \\ &\cong \mathrm{M}_n(A) \text{ from [3, lemma2.1] }.\blacksquare \end{aligned}$$

Lemma 3.8

$$(\mathrm{M}_n(A,M))_d \cong \mathrm{M}_n(A_d,M)$$

Proof

Since A_d is embeded in A then $\mathrm{M}_n(A_d,M)$ is embeded in $\mathrm{M}_n(A,M)$ i.e

$$\mathrm{M}_n(A_d,M) \hookrightarrow \mathrm{M}_n(A,M)$$

so

$$(\mathrm{M}_n(A_d,M))_d \hookrightarrow (\mathrm{M}_n(A,M))_d \quad .$$

But $\mathrm{M}_n(A_d,M)$ is distributive so $(\mathrm{M}_n(A_d,M))_d$ $=\mathrm{M}_n(A_d,M)$ which implies that

$$\mathrm{M}_n(A_d,M) \hookrightarrow (\mathrm{M}_n(A,M))_d \ .$$

But there is a one to one correspondence betwean the set of generators of $(\mathrm{Mat}_n(A,M))_d$ and the set of generators of $\mathrm{Mat}_n(A_d,M)$ becouse $(\mathrm{M}_n(A,M))_d$ is embeded in the the zero-symmetric part of $\mathrm{M}_n(A,M)$ and we have

$$(f_{ij}^r,0) \in (\mathrm{M}_n(A,M))_d \text{ iff } r \in A_d.$$

So

$$(\mathrm{Mat}_n(A,M))_d \cong \mathrm{Mat}_n(A_d,M).\blacksquare$$

We use lemma 3.8 to proof the following theorem which gives a necessary and sufficient condition for which $\mathrm{Mat}_n(A,M)$ is an abstract affine near ring.

Theorem 3.9

$\mathrm{M}_n(A_0{*}A_c,N)$ is abstract affine near ring (a.a.n.r for short) iff N is a ring A_0-module

Proof

$(\Rightarrow)$ Let $\mathrm{M}_n(A_0*A_c,N)$ is an a.a.n.r then we have $\mathrm{M}_n(A_0*A_c,N)$ is abelian and so N is abelian (1). We show that

$$\forall r \in A_0, \forall l,m \in N \quad r(l+m) = rl + rm.$$

Since$\mathrm{M}_n(A_0{*}A_c,N)$ is an a.a.n.r then $(\mathrm{M}_n(A_0{*}A_c,N)))_0 = (\mathrm{Mat}_n(A_0{*}A_c,N))_d$ and so we have if $r \in A_0, 1 \le i,j,k,h \le n$

$$(f_{ij}^r,0)((f_{jk}^1,0)+(f_{jh}^1,0)) = (f_{ij}^r,0)(f_{jk}^1,0)+(f_{ij}^r,0)(f_{jh}^1,0)$$

so if $n_k = m, n_h = l \in N$ then

$$[(f_{ij}^r,0)((f_{jk}^1,0)+(f_{jh}^1,0)](0,...,m,0,...,l,0,...,0) =$$

$$[(f_{ij}^r,0)(f_{jk}^1,0)+(f_{ij}^r,0)(f_{jh}^1,0)](0,...,m,0,...,l,0,...,0)$$

so

$$(f_{ij}^r,0)(0,...,m+l,0,...,0) = (f_{ij}^r,0)(0,...,m,0,...,0)+(f_{ij}^r,0)(0,...,l,0,...,0)$$

then

$$r(m+l) = rm + rl \quad (2)$$

so from (1),(2) we have N is a ring A_0-module.
($\Leftarrow$) Let N be a ring A_0-module we have $\mathrm{M}_n(A_0 * A_c, N)$ is the direct sum of $\mathrm{M}_n(A_0 \oplus \{0\}, N), A_c^n$ i.e

$$\mathrm{M}_n(A_0 * A_c, N) = \mathrm{M}_n(A_0, N) \oplus A_c^n.$$

Since A_0, N are abelian so $\mathrm{M}_n(A_0, N)$ is abelian also A_c^n is abelian since A_c is a belian then $\mathrm{M}_n(A_0 * A_c, N)$ is abelian (1). Also from lemma 3.8 we have

$$\begin{aligned}(\mathrm{M}_n(A_0 * A_c, N))_d &\cong \mathrm{M}_n((A_0 * A_c)_d, N)) \\ &= \mathrm{M}_n(A_0, N) \quad .(2)\end{aligned}$$

$\mathrm{M}_n(A_0 * A_c, N)$ is a.a.n.r.■

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