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Using the Simplified Hirota's Method to Investigate Multi-Soliton Solutions of the Fifth-Order KdV Equation

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Abstract

In this work we use the simplified Hirota's method to investigate the multi-soliton solutions for the fifth-order KdV equation. The two-soliton and three-soliton solutions are obtained. The dynamic properties of these multi-soliton solutions discussed. the 3-D and 2-D graphs of the two-soliton and three-soliton solutions are given. The process of the reciprocity between the soliton and soliton are shown by the profiles.

Keywords: Hirota's method, Simplified Hirota's method, Simplied version of Hirota's method, Fifth-order KdV equation, Multi-soliton solutions

1 Introduction

Seeking the Multi-soliton solutions of the nonlinear wave equations (NWEs) or PDEs plays an important role in treating nonlinear problems, and many powerful methods have been proposed and developed. Due to the availability of symbol-calculation systems of computer, more attention is paid to simple and direct methods which allow one to construct explicit specific solutions for NWEs or PDEs. The Hirota method is the direct method which is most frequently used to construct soliton solutions of nonlinear PDEs from soliton theory (and beyond)[1] . As is well known, the Hirota method is applicable to nonlinear equations which take a bilinear form, if the bilinear form exists at all, and is highly nontrivial. In order to overcome this difficulty, similar to [2,3] we choose the simplified version of Hirota's method to show how to combine the homogeneous balance principle with the simplified Hirota method for seeking multi-soliton solutions with the aid of Maple. As an example, we will investigate multi-soliton solutions of the following fifth-order KdV equation [4-8]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x. \quad (1)$$

2 The brief introduction of simplified Hirota's method

In order to apply the simplified Hirota's method for finding multi-soliton solutions, the key step is to derive a nonlinear transformation which transforms the original equation into its bilinear form. Here we introduce the simplified version of Hirota's method to show how to combine the homogeneous balance principle with the simplified Hirota method for seeking multi-soliton solutions with the aid of Maple.

Step 1. For a non-linear partial differential equations

$$P(u, u_x, u_t, u_{xt}, u_{xx}, u_{tt}, \cdots) = 0, \quad (2)$$

Step 2. A non-linear transformation is introduced as follows:

$$u = f(w), \quad (3)$$

where $w = w(x,t)$. Substituting (3) into (2) yields

$$Q(w) = 0. \quad (4)$$

Step 3. According to coefficients of every power for w, we introduce the linear operator $L\bullet$ and non-linear operator$N_1\bullet, N_2\bullet, N_3\bullet, \cdots$.

Step 4. Suppose that

$$w(x,t) = 1 + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \cdots + \epsilon^r w_r + \cdots, \quad (5)$$

substituting (5) into (4), we sort order cording to the power of ϵ.

Step 5. Let the coefficient of each power of ϵ as zero, to get the recursive equations for $w_1, w_2, w_3, \cdots w_r, \cdots$ as follows:

$$\epsilon: \quad L \bullet w_1 = 0, \quad (6)$$

$$\epsilon^2: \quad \text{Ł} \bullet w_2 = N_1 \bullet (w_1, w_1) + \cdots \quad (7)$$

$$\epsilon^3: \quad \text{Ł} \bullet w_3 = N_2 \bullet (w_1, w_2) + \cdots \quad (8)$$

$$\cdots\cdots$$

Among the above iterative equations, from the Eq. (7) onwards, from the top down, every right expression which is in these recursive equations depends on the solutions of the previous equations. In the iterative process, we will obtain a series of solutions such as $w_1,\ w_2,\ w_3,\ \cdots$. Among these solutions, if one of solution $w_i = 0$ is happened, then from this expression downward all the solutions $w_k = 0$ of the posterior equations as long as $k > i$. Thus, the Eq. (5) has been truncated, then the solutions of Eq. (2) can be obtained.

3 Multi-soliton solution of fifth-order KdV equation

In this section, we will investigate multi-soliton solutions of Eq. (1) by the introduced method in the section 2. Next, we suppose that the bilinear transformation of (1) as follows:

$$u(x,t) = M[ln(w)]_{xx}, \tag{9}$$

where $w = 1 + e^{kx-vt}$.Substituting the Eq. (9) into Eq. (1), it is easy to know $M = 2$. Thus the transformation (9) becomes the following equation

$$u(x,t) = 2[ln(w)]_{xx} \tag{10}$$

where $w = w(x,t)$. Suppose that

$$w(x,t) = 1 + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \cdots + \epsilon^r w_r + \cdots, \tag{11}$$

where w_r $(r = 1, 2, 3, ...)$ are functions on x, t, ϵ is a small parameter.

Substituting the Eq. (10) into Eq. (1), Eqs. (1) becomes the following homogeneous differential equation for the function w and its derivatives for every order which are shown the below:

$$\begin{aligned} w^3(w_{xxt} - w_{xxxxxxx}) + w^2(-w_{xx}w_t - 2w_x w_{xt} + 7w_{xxxxxx}w_x + w_{xxxxx}w_{xx} - 5w_{xxxx}w_{xxx}) \\ +w(2w_x^2 w_t + 10w_{xxxx}w_x w_{xx} - 10w_{xxx}w_{xx}^2 + 20w_{xxx}^2 w_x - 22w_{xxxxx}w_x^2) \\ +30w_{xxxx}w_x^3 - 60w_{xxx}w_x^2 w_{xx} + 30w_{xx}^3 w_x = 0. \end{aligned} \tag{12}$$

The Eq. (12) can be rewritten as

$$w^3 L \bullet w + w^2 N_1 + wN_2 + N_3 = 0, \tag{13}$$

where L is linear operator, the $L,\ N_1,\ N_2,\ N_3$ are defined as follows
$L\bullet = \frac{\partial^3}{\partial x^2 \partial t} - \frac{\partial^7}{\partial x^7}$
$N_1(f,g) = -f_{xx}g_t - 2f_x g_{xt} + 7f_{xxxxxx}g_x + f_{xxxxx}g_{xx} - 5f_{xxxx}g_{xxx}$
$N_2(f,g,h) = 2f_x g_x h_t + 10f_{xxxx}g_x h_{xx} - 10f_{xxx}g_{xx}h_{xx} + 20f_{xxx}g_{xxx}h_x - 22f_{xxxxx}g_x h_x$

$$N_3(f,g,h,k) = 30f_{xxxx}g_x h_x k_x - 60f_{xxx}g_x h_x k_{xx} + 30f_{xx}g_{xx}h_{xx}k_x, \tag{14}$$

and f, g, h, k are auxiliary functions.

In order to obtain (1)'s multi-soliton solutions, substituting the Eq. (11) into Eq. (12) and by using the step 5 in section 2, we obtain a series of iterative equations for w_i as follows:

$$\varepsilon : L\bullet w_1 = 0, \tag{15}$$

$$\varepsilon^2 : L \bullet w_2 = -N_1(w_1, w_1) - 3w_1(L \bullet w_1), \tag{16}$$

$$\begin{aligned}\varepsilon^3 : L \bullet w_3 = &-(3w_2 + 3w_1^2)L \bullet w_1 - N_1(w_2, w_1) - N_2(w_1, w_1, w_1) - N_1(w_1, w_2)\\ &-3w_1 L \bullet w_2 - 2w_1 N_1(w_1, w_1),\end{aligned} \tag{17}$$

$$\begin{aligned}\varepsilon^4 : L \bullet w_4 = &-N_3(w_1, w_1, w_1, w_1) - N_1(w_3, w_1) - N_2(w_2, w_1, w_1) - N_2(w_1, w_1.w_2)\\ &-N_1(w_1, w_3) - N_2(w_1, w_2, w_1) - N_1(w_2, w_2) - 6(L \bullet w_1)w_1 w_2 - 2N_1(w_1, w_1)w_2\\ &-3(L \bullet w_2)w_2 - 2w_1 N_1(w_1, w_2) - 3w_1 L \bullet w_3 - 2w_1 N_1(w_2, w_1) - N_1(w_1, w_1)w_1^2\\ &-w_1 N_2(w_1, w_1, w_1) - 3(L \bullet w_2)w_1^2 - 3(L \bullet w_1)w_3 - (L \bullet w_1)w_1^3,\end{aligned} \tag{18}$$

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First consider (1)'s one-soliton solutions, suppose that

$$w_1 = e^{\theta}, \quad \theta = kx - vt, \tag{19}$$

where k, v are constants. Substituting the Eq. (19) into Eq. (15), we obtain a dispersion relationship $v = -k^5$, where k is an arbitrary constant. Thus we have

$$w_1 = e^{\theta}, \quad \theta = kx + k^5 x. \tag{20}$$

Substituting the Eq. (20) into Eq. (16) yields

$$L \bullet w_2 = 0. \tag{21}$$

It is easy to know $w_2 = 0$. From this expression downward, all the solutions $w_r = 0$ of the posterior equations as long as $r > 2$. Without loss of generality, let $\varepsilon = 1$. By using (20),(11),(10), we obtain one-soliton solution of Eq. (1) as follows:

$$u(x, t) = \frac{2k^2 e^{\theta}}{(1 + e^{\theta})^2}, \tag{22}$$

where $\theta = kx + k^5 t$.

Second, we consider the double-soliton solutions of Eq. (1). Suppose that

$$w_1 = e^{\theta_1} + e^{\theta_2}, \tag{23}$$

where $\theta_i = k_i x - v_i x \quad (i = 1, 2)$, the k_i, v_i are constants.

Substituting the Eq.(23) into Eq. (15), we obtain dispersion relationship as follows:

$$v_1 = -k_1^5, \quad v_2 = -k_2^5. \tag{24}$$

Substituting the Eqs. (23) and (24) into Eq. (16) yields

$$L \bullet w_2 = -5k_1 k_2 (k_1^2 - k_2^2)(k_1^3 - k_2^3)e^{\theta_1 + \theta_2}. \tag{25}$$

Solving the above linear equation (25), we obtain

$$w_2 = a_{12} e^{\theta_1+\theta_2}, \tag{26}$$

where $a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$. Substituting w_1, w_2 into Eq. (17), we obtain

$$L \bullet w_3 = 0. \tag{27}$$

Thus we have $w_3 = 0$. Substituting the w_1, w_2, w_3 into the posterior iterative equations, we easily obtain $w_r = 0$ as long as $r \geq 3$. Taking $\varepsilon = 1$, then by using (23), (26), (11) and (10), we obtain a double-soliton solution of Eq. (1) as follows

$$u = \frac{2[k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + (k_1-k_2)^2 e^{\theta_1+\theta_2}]}{1 + e^{\theta_1} + e^{\theta_2} + \frac{(k_1-k_2)^2 e^{\theta_1+\theta_2}}{(k_1+k_2)^2}} - \frac{2[k_1 e^{\theta_1} + k_2 e^{\theta_2} + \frac{(k_1-k_2)^2 e^{\theta_1+\theta_2}}{(k_1+k_2)}]^2}{[1 + e^{\theta_1} + e^{\theta_2} + \frac{(k_1-k_2)^2 e^{\theta_1+\theta_2}}{(k_1+k_2)^2}]^2}, \tag{28}$$

where $\theta_i = k_i x + k_i^5 t,\ i = 1, 2$.

At last, we consider three-soliton solutions of Eq. (1). Suppose that

$$w_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \quad \theta_i = k_i x - v_i x, \tag{29}$$

where k_i, v_i $(i = 1, 2, 3)$ need to be determined further. Substituting (29) into (15) yields

$$v_1 = -k_1^5, \quad v_2 = -k_2^5, \quad v_3 = -k_3^5. \tag{30}$$

Substituting (29) and (30) into (16) yields

$$L \bullet w_2 = -5k_1k_2(k_1^2 - k_2^2)(k_1^3 - k_2^3)e^{\theta_1+\theta_2} - 5k_1k_3(k_1^2 - k_3^2)(k_1^3 - k_3^3)e^{\theta_1+\theta_3}$$
$$-5k_2k_3(k_2^2 - k_3^2)(k_2^3 - k_3^3)e^{\theta_2+\theta_3}. \tag{31}$$

Solving the Eq. (31), we obtain

$$w_2 = a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3}, \tag{32}$$

where $a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}$ $(1 \leq i \leq j \leq 3)$.

Substituting w_1, w_2 into (17), we have

$$w_3 = a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}. \tag{33}$$

Substituting w_1, w_2, w_3 into Eq. (18) yields

$$L \bullet w_4 = 0. \tag{34}$$

Thus we obtain $w_4 = 0$. From this expression downward, all the solutions $w_r = 0$ of the posterior equations as long as $r > 4$. Without loss of generality, let $\varepsilon = 1$. By using Eqs. (29), (30), (32) ,(33)and (11), we obtain

$$w = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{123}e^{\theta_1+\theta_2+\theta_3}, \tag{35}$$

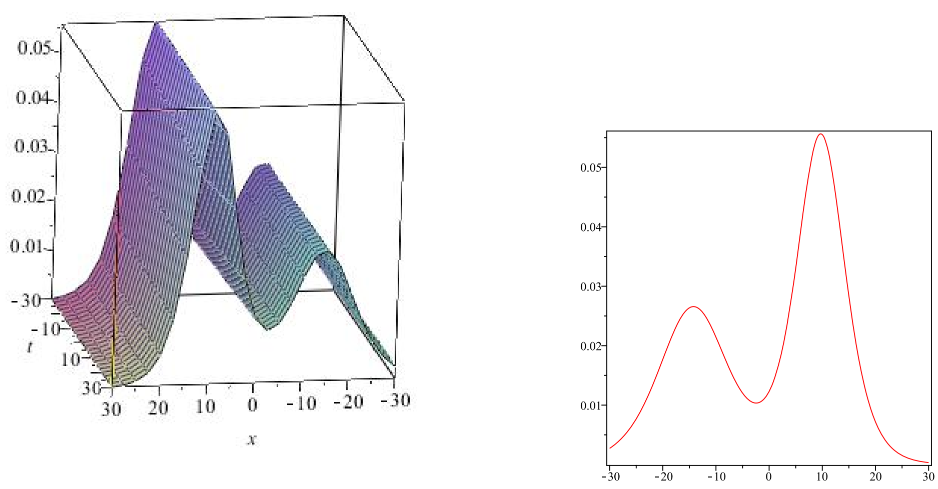
where $a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}$ $(1 \leq i \leq j \leq 3)$, $a_{123} = a_{12}a_{13}a_{23}, \theta_i = k_i x + k_i^5 t$ $(i = 1, 2, 3)$. By using Eqs. (35) and (10), we can always obtain three-soliton solutions of Eq. (1) as follows:

$$u = 2[\ln(1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+a_{123}e^{\theta_1+\theta_2+\theta_3})]_{xx}, \tag{36}$$

where a_{ij}, a_{123}, θ_i $(i = 1, 2, 3)$ are given above.

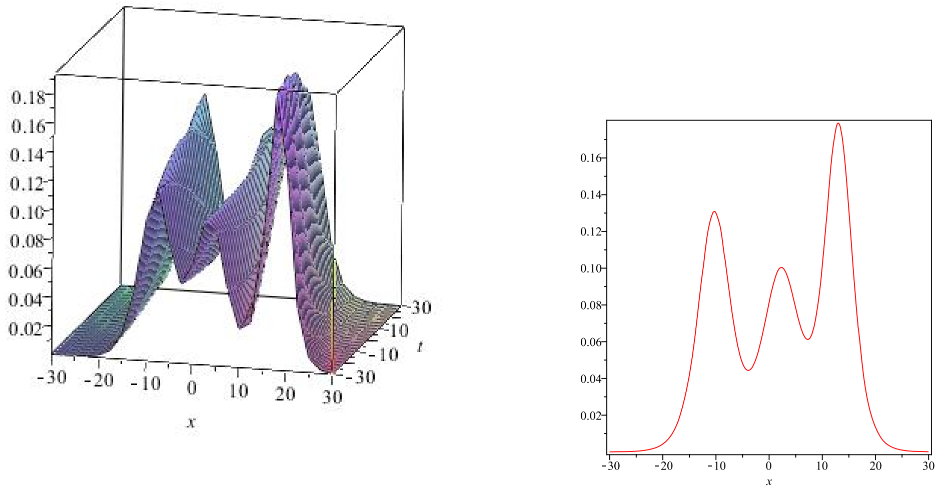
The N-soliton solution for any $N > 3$ can be always constructed in a similar way. However, the calculations become very lengthy, so we omit these results here.

In order to show the dynamic properties of above double soliton and three-soliton solutions intuitively, as examples, we plot the 3D and 2D-graphs of double solitonsolutions (28), see Fig.1; the the 3D and 2D-graphs of three-sc



(a) (b)

Fig. 1. The 3D and 2D graphs of two-soliton solutions.



(c) (d)

Fig. 2. The 3D and 2D graphs of three-soliton solutions.

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References

[1] Hirota R., Direct methods in soliton theory, in solitons [A]. eds. R. K. Bullough and P. J. Caudrey, Topics of Modern Physics[C] . Springer, New York, 1980.

[2] Willy H., Ameina N., Symbolic methods to construct exact solutions of nonlinear partial differential equations, Mathematics and Computers in Simulation 43 (1997) 13-27.

[3] A.-M. Wazwaz, Multiple-soliton solutions for the Boussinesq equation, Applied Mathematics and Computation 192 (2007) 479-486.

[4] Rafei, M., Ganji, D. D., Explicit Solutions of Helmholtz Equation and Fifth-order KdV Equation using Homotopy Perturbation Method, International Journal of Nonlinear Sciences and Numerical Simulation 7 (2006) 321-328.

[5] A.-M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, Applied Mathematics and Computation 184 (2007) 1002-1014.

[6] D. Kaya, An explicit and numerical solutions of some fifth-order KdV equation by decomposition method, Applied Mathematics and Computation 144 (2003) 353-363.

[7] S. Abbasbandy and F. Samadian Zakaria, Soliton solutions for the fifth-order KdV equation with the homotopy analysis method, Nonlinear Dynamics 51 (2008) 83-87.

[8] D. Kayaa, Salah M. El-Sayed, On a generalized fifth order KdV equations, Physics Letters A 310 (2003) 44-51.

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