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Multigrid Method for Solving 2D-Helmholtz Equation with Sixth Order Accurate Compact Finite Difference Method

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Abstract

In this paper we develop a sixth order finite difference discretization strategy to solve the two dimensional Helmholtz equation. We use multigrid V-cycle procedure to built multiscale multigrid method which is similar to the full multigrid method. Numerical result is given to illustrate this method.

1-Introduction

Poisson equation is a partial differential equation (PDF) with broad application s in mechanical engineering, theoretical physics and other fields. The two dimensional (2D) Poisson equation can be written in the form:

$$u_{xx}(x,y)+u_{yy}(x,y)+k^2u(x,y)=f(x,y), \quad (x,y)\in\Omega \tag{1}$$

Where Ω is a rectangular domain, or union of rectangular domains, with suitable boundary conditions defined on $\partial\Omega$. The solution *u(x,y)* and forcing function *f(x,y)* are assumed to be sufficiently smooth and have the necessary derivatives up to certain orders.

A second order accurate solution can be computed by applying standard second order centeral difference operators $u_{xx}(x,y)$ and $u_{yy}(x,y)$ in Eq. (1). Higher order (more than two) accurate discertization methods need more complex procedure than the second order accurate discretization method to compute the coefficient matrix, but they usually generate linear systems of much smaller size, compared with that from the lower order accurate discetization method [1,4] . There has been growing interest in developing higher order accurate discretization methods, especially the high order compact difference scheme to solve partial differential equations (PDEs) [6, 8, 9, and 10]

2-Sixth Order Compact Approximation

This method is similar to a sixth-order accurate approximation to the derivatives calculated from Helmholtz equation [5]. We the scheme for the two dimensional uniform Cartesian grids with grid spacing $\Delta x=\Delta y=h.$ the mesh points are (x_i, y_j) with $x_i = ih$ a $y_j = jh, \quad 0\le i\le N, \; 0\le j\le N$ where N is the number of uniform intervals in the x and y directions.

We write the second order central difference operators as

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \qquad \delta_y^2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}, \tag{2}$$

Using Taylor series expansions at the grid point (x_i, y_j), we have

$$\delta_x^2 u_{i,j} = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{h^6}{360}\frac{\partial^6 u}{\partial x^6} + O(h^6) \tag{3}$$

$$\delta_y^2 u_{i,j} = \frac{\partial^2 u}{\partial y^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4} + \frac{h^6}{360}\frac{\partial^6 u}{\partial y^6} + O(h^6) \tag{4}$$

The central finite difference for Eq. (1) can be written as

$$\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} + k^2 u_{i,j} + T_{i,j} = f_{i,j} \tag{5}$$

Where

$$u_{i,j} = u(x_i, y_j),\ f_{i,j} = f(x_i, y_j)$$

and

$$T_{i,j} = -\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) - \frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right) + O(h^6) \tag{6}$$

Using the following appropriate derivatives of Eq. (1)

$$\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} - k^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^2 \partial^2 y}, \qquad \frac{\partial^4 u}{\partial y^4} = \frac{\partial^2 f}{\partial y^2} - k^2 \frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial y^2 \partial x^2}, \tag{7}$$

In Eq. (6) we get

$$T_{i,j} = -\frac{h^2}{12}\left(\nabla^2 f_{i,j} - 2\left[\frac{\partial^4 u}{\partial x^2 \partial y^2}\right]_{i,j} - k^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]_{i,j}\right) - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right]_{i,j} + O(h^6) \tag{8}$$

The fourth order approximation of $\dfrac{\partial^4 u}{\partial x^2 \partial^2 y}$ in Eq. (8) can be written as

$$\left[\frac{\partial^4 u}{\partial x^2 \partial^2 y}\right]_{i,j} = \delta_x^2 \delta_y^2 u_{i,j} - \frac{h^2}{12}\left[\frac{\partial^6 u}{\partial x^4 \partial^2 y} + \frac{\partial^6 u}{\partial x^2 \partial^4 y}\right]_{i,j} + O(h^4) \tag{9}$$

Substituting Eq. (9) into Eq. (8), we get

$$T_{i,j} = -\frac{h^2}{12}\left(\nabla^2 f_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j} + k^2 f_{i,j} - k^4 u_{i,j}\right) - \frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6} + 5\frac{\partial^6 u}{\partial x^4 \partial^2 y} + 5\frac{\partial^6 u}{\partial x^2 \partial^4 y} + \frac{\partial^6 u}{\partial y^6}\right)_{i,j} + O(h^6) \tag{10}$$

Clearly, getting a compact sixth-order approximation requires compact expressions of the four Derivatives of order six in Eq. (1) which can be done by further differentiating Eq. (10) that is

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4} + k^2 \frac{\partial^4 u}{\partial x^2 \partial y^2} \tag{11}$$

$$\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} = \nabla^4 f - k^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) - \left(\frac{\partial^6 u}{\partial x^4 \partial y^2} + \frac{\partial^6 u}{\partial x^2 \partial y^4}\right) \tag{12}$$

Substituting Eq. (7), (11) into Eq. (13) gives

$$\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6} = \nabla^4 f - \frac{\partial^4 f}{\partial x^2 \partial y^2} - k^2\nabla^2 f + k^4(-k^2 u + f) + 3k^2 \frac{\partial^4 f}{\partial x^2 \partial y^2} \tag{13}$$

Using Eq. (11), (13) we can eliminate all derivatives of u in Eq. (10) that is

$$T_{i,j} = \frac{h^2}{12}\left(-\nabla^2 f_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j} + k^2 f_{i,j} - k^4 u_{i,j}\right) - \frac{h^4}{360}\left(\nabla^4 f_{i,j} + 4\left[\frac{\partial^4 u}{\partial x^2 \partial^2 y}\right]_{i,j} - k^2\nabla^2 f_{i,j} + k^4 f_{i,j} - k^2 u_{i,j} - 2k^2\delta_x^2\delta_y^2 u_{i,j}\right)_{i,j} + O(h^6) \tag{14}$$

The compact sixth-order approximation of two dimensional can be written as

$$\frac{h^2}{6}\left(1+\frac{k^2h^2}{30}\right)\delta_x^2\delta_y^2 u_{i,j}+\left(\delta_x^2+\delta_y^2\right)u_{i,j}+k^2\left(1-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right)u_{i,j}$$
$$=\left(1-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right)f_{i,j}+\left(\frac{h^2}{12}\left(1-\frac{k^2h^2}{30}\right)\right)\frac{h^2}{12}\nabla^2 f_{i,j}+\frac{h^4}{360}\nabla^4 f_{i,j}+\frac{h^4}{90}\left[\frac{\partial^4 f}{\partial_x^2\partial_y^2}\right] \quad (15)$$

We can express Eq. (16) in the form

$$d_{20}u_{i,j}+d_{21}\left(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right)$$
$$+d_{22}\left(u_{i+1,j+1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i-1,j-1}\right)$$
$$=b_{20}f_{i,j}+b_{21}\left(f_{i+1,j}+f_{i-1,j}+f_{i,j+1}+f_{i,j-1}\right)$$
$$+b_{22}\left(f_{i+1,j+1}+f_{i+1,j-1}+f_{i-1,j+1}+f_{i-1,j-1}\right)+\frac{h^6}{90}\left[\frac{\partial^4 f}{\partial_x^2\partial_y^2}\right] \quad (16)$$

Where

$$d_{20}=-\frac{10}{3}+k^2h^2\left(\frac{46}{45}-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right),\quad d_{21}=\frac{2}{3}-\frac{k^2h^2}{90},\quad d_{22}=\frac{1}{6}+\frac{k^2h^2}{180}$$
$$b_{20}=h^2\left(1-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right),\quad b_{21}=\frac{h^4}{12}\left(1-\frac{k^2h^2}{30}\right),\quad b_{22}=\frac{h^6}{360},\quad b_{23}=\frac{h^6}{90} \quad (17)$$

3- Multigrid Method V-Cycle

Algorithm

1- Let u_{2h} be the solution on the coarse grid Ω_{2h}
2- Use some high order interpolation schemes here we use Newton difference interpolation, to interpolate $\Omega_{2h}, u_h = I_{2h}^{h}$ to the coarse grid Update every (odd-odd) grid point on

From Eq. (16) for each point *(imp)* the updated solution is

$$u_{i,j}^{k+1} = \left[\begin{array}{l} F_{i,j} - d_{21}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) \\ - d_{22}\left(u_{i+1,j+1}^{k} + u_{i-1,j+1}^{k} + u_{i+1,j-1}^{k} + u_{i-1,j-1}^{k}\right) \end{array} \right] / d_{20} \quad (18)$$

Here, $F_{i,j}$ represents the right-hand side of Eq. (16)

3- Update every (odd, even) grid point on Ω_h from Eq. (16)

4- Update every (even, odd) grid point on Ω_h from Eq. (16)

5- Compute the l_2 norm Ω_{2h} $R = \left\| u^{h,k+1} - u^{h,k} \right\|_2$ if not converged go back to step 3.

6- Compute residual on the fine grid Ω_h from $L_h - f_h = d_h$ and use full weighting scheme to project residual from fine grid to the coarse grid.

7- Use interpolation to transfer corrections from the coarse grid to the fine grid.

8- Relax υ_1 times on $L_h u_h = f_h$.

9- Use from the previous step as the initial guess to run the multigrid algorithm until it converges.

4- Numerical Results

Problem A Consider Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = (k^2 - 2\pi^2)\sin(\pi x)\sin(\pi y) \quad 0 \le x \le 1 \text{ And } 0 \le y \le 1$$

With Dirichlet boundary conditions on all sides f unit square, that is

$$u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0$$

The exact solution is $u(x,y) = \sin \pi x \sin \pi y$

In order to compare the numerical solution to the exact solution we use two performance metrics namely e is defined as

$$\|e\|_2 = \frac{1}{N}\sqrt{\sum_{i,j=0}^{N} e_{i,j}^2} \tag{19}$$

Where $e_{i,j}$ =numerical solution-exact solution

The metric order is defined as

$$Order(n, n+1) = \log_2 \frac{\|e\|_\infty (n)}{\|e\|_\infty (n+1)} \tag{20}$$

And measures the order of numerical $\|e\|_\infty$ in Eq. (18) is called l_∞ - norm of the error vector and is defined as

$$\|e\|_\infty = \max_{0 \le i,j \le N} e_{i,j} \tag{21}$$

The l_2 - norm of the error and the order of accuracy, for different values of N are presented in Table 1.

N	$\|e\|_\infty$	$\|e\|_2$	Order
2	0.21850903e-1	0.72836343e-2	
4	0.27610356e-3	0.11044145e-3	6.04
8	0.40788436e-5	0.18128598e-5	5.9289
16	0.6299437e-7	0.29689777e-7	5.9322
32	0.11497659e-8	0.61250031e-8	5.5991
64	0.39523036e-9	0.18549957e-9	5.0452
128	0.40222975e-10	0.13368717e-10	3.7945
256	not converges	not converges	

Table 1

6- Conclusion

A sixth-order accurate compact finite difference scheme with multigrid method for solving Helmoltz equation was developed. Numerical computation showing the efficiency of this method.

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