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IFSM's of an R-Module - A Study

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Abstract

As a generalization of fuzzy sets,the concept of intuitionistic fuzzy sets was introduced by Krassimir T. Atanassov in 1986. In this article, we study about the support and level sets of intuitionistic fuzzy sets in an R-module. Also we study some properties of intuitionistic fuzzy submodules (IFSM's) of an R-module and prove some characterizations for IFSM's.

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1 Introduction.

As a generalization of a fuzzy set, the concept of an intuitionistic fuzzy set was introduced by K.T. Atanassov [1, 2]. Applying this concept to algebra, B. Davvas etal.[4] established the intuitionistic fuzzification of the concept of submodules of an R-module. In this paper, in section 2 we give the essential preliminaries and in section 3 we study about the properties of the support and level sets of intuitionistic fuzzy sets in an R-module M. In section 4 we provide a necessary and sufficient condition for an intuitionistic fuzzy subset in an R-module M to be an intuitionistic fuzzy submodule of M.

Throughout this paper, we denote by I the unit interval $[0, 1]$, by R a commutative ring with unity 1 and by M a unitary R-module. $\vee$ denotes the maximum, and $\wedge$ the minimum in the unit interval $[0, 1]$.

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2 Preliminaries

In this section we give some basic definitions and results which are used in the sequel. For knowledge regarding modules and fuzzy modules we refer the books by Hungerford [5] and Mordeson & Malik [8] respectively.

2.1. Definition ([1]). An intuitionistic fuzzy set (in short IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x))/x \in X\}$ where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we will denote the set of all IFS's in X as IFS(X).

2.2. Definition ([1]). Let X be a non-empty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be IFS's in X. Then

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$
2. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$
3. $A^C = (\nu_A, \mu_A)$
4. $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)) : x \in X\}$
5. $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) : x \in X\}$

2.3. Definition ([10]). A fuzzy set μ in M is called a fuzzy submodule of M if for every $x, y \in M$ and $r \in R$, the following conditions are satisfied

1. $\mu(0) = 1$
2. $\mu(x + y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(rx) \geq \mu(x)$

2.4. Definition ([4]). Let M be a module over a ring R. An IFS $A = (\mu_A, \nu_A)$ in M is called an intuitionistic fuzzy submodule (IFSM) of M if

1. $\mu_A(0) = 1 \quad \text{and} \quad \nu_A(0) = 0$
2. $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)\ \forall x, y \in M$
3. $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)\ \forall x, y \in M$
4. $\mu_A(rx) \geq \mu_A(x)\ \forall x \in M, \forall r \in R$
5. $\nu_A(rx) \leq \nu_A(x)\ \forall x \in M, \forall r \in R$

Remark. By saying that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy module (IFM) we mean that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy submodule of some R module M.

2.5. Definition ([3]). Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFS's in M. Then we define their sum $A+B$ as the IFS $A+B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x)) : x \in M\}$ where for each $x \in M$,

$$\begin{aligned} \mu_{A+B}(x) &= \vee\{\mu_A(y) \wedge \mu_B(z) : y, z \in M, x = y+z\}, \text{ and} \\ \nu_{A+B}(x) &= \wedge\{\nu_A(y) \vee \nu_B(z) : y, z \in M, x = y+z\} \end{aligned}$$

2.6. Definition ([7]). For an IFS $A = (\mu_A, \nu_A)$ in M and for any $r \in R$, we define the IFS $rA = (\mu_{rA}, \nu_{rA})$ in M as $rA = \{(x, \mu_{rA}(x), \nu_{rA}(x)) : x \in M\}$ where for each $x \in M$, $\mu_{rA}(x) = \vee\{\mu_A(y) : y \in M, x = ry\}$, and $\nu_{rA}(x) = \wedge\{\nu_A(y) : y \in M, x = ry\}$.

3 Support and Level Sets

In this section we study about the properties of the support and level sets of intuitionistic fuzzy sets in an R-module M.

3.1. Definition. Let X be a non-empty set and let $A = (\mu_A, \nu_A)$ be an IFS in X. Then the support of A, denoted by A^*, is defined as $A^* = \{x \in X : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}$.

3.2. Theorem. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFS's in an R-module M. If $A \subseteq B$, then $A^* \subseteq B^*$.*

Proof. Since $A \subseteq B$, we have $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall x \in M$. Suppose $x \in A^*$. Then $\mu_A(x) > 0$ and $\nu_A(x) < 1$. So we get $\mu_B(x) \geq \mu_A(x) > 0$ and $\nu_B(x) \leq \nu_A(x) < 1$, and hence $x \in B^*$. Therefore $A^* \subseteq B^*$.

3.3. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFSM of an R-module M. Then A^* is a submodule of M.*

Proof. Suppose $x, y \in A^*$. Then $\mu_A(x) > 0, \mu_A(y) > 0$ and $\nu_A(x) < 1, \nu_A(y) < 1$. Now $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y) > 0$ and $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y) < 1$. So $x - y \in A^*$. Now for $x \in A^*$ and $r \in R$, $\mu_A(rx) \geq \mu_A(x) > 0$ and $\nu_A(rx) \leq \nu_A(x) < 1$. So we get $rx \in A^*$. Hence by definition, A^* is a submodule of M.

3.4. Definition ([6]). Let X be a nonempty set and $A = (\mu_A, \nu_A)$ be an IFS in X, and $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$. Then the (α, β)-level set of A is defined as $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$.

Now we state without proof the following theorem:

3.5. Theorem. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFS's in a nonempty set X, then we have the following:*

1. $A \subseteq B$, $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \leq 1 \Rightarrow A_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$.
2. $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ *with* $\alpha_1 + \beta_1 \leq 1$, $\alpha_2 + \beta_2 \leq 1$, $\alpha_1 \leq \alpha_2$ *and* $\beta_1 \geq \beta_2 \Longrightarrow A_{(\alpha_2,\beta_2)} \subseteq A_{(\alpha_1,\beta_1)}$.
3. $A = B \Longleftrightarrow A_{(\alpha,\beta)} = B_{(\alpha,\beta)}$ $\forall \alpha, \beta \in [0,1]$ *with* $\alpha + \beta \leq 1$.

3.6. Theorem. *Let $A_i = (\mu_{A_i}, \nu_{A_i})$; $i \in J, |J| > 1$, be a family of IFS's in X. Then for any $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, the following hold:*

1. $\bigcup_{i\in J} (A_i)_{(\alpha,\beta)} \subseteq (\bigcup_{i\in J} A_i)_{(\alpha,\beta)}$
2. $\bigcap_{i\in J} (A_i)_{(\alpha,\beta)} = (\bigcap_{i\in J} A_i)_{(\alpha,\beta)}$

Proof.

(i). We have,
$$\begin{aligned} x \in \bigcup_{i\in J} (A_i)_{(\alpha,\beta)} &\Rightarrow x \in A_{j_{(\alpha,\beta)}} \text{ forsome } j \in J \\ &\Rightarrow \mu_{A_j}(x) \geq \alpha,\ \nu_{A_j}(x) \leq \beta \\ &\Rightarrow \bigvee_{i\in J} \mu_{A_i}(x) \geq \alpha,\ \bigwedge_{i\in J} \nu_{A_i}(x) \leq \beta \\ &\Rightarrow \mu_{\underset{i\in J}{\cup} A_i}(x) \geq \alpha,\ \nu_{\underset{i\in J}{\cup} A_i}(x) \leq \beta \\ &\Rightarrow x \in (\bigcup_{i\in J} A_i)_{(\alpha,\beta)} \end{aligned}$$

(ii). Also,
$$\begin{aligned} x \in \bigcap_{i\in J} (A_i)_{(\alpha,\beta)} &\Leftrightarrow x \in A_{i(\alpha,\beta)}\ \ \forall\, i \in J \\ &\Leftrightarrow \mu_{A_i}(x) \geq \alpha,\ \nu_{A_i}(x) \leq \beta,\ \ \forall\, i \in J \\ &\Leftrightarrow \bigwedge_{i\in J} \mu_{A_i}(x) \geq \alpha,\ \bigvee_{i\in J} \nu_{A_i}(x) \leq \beta \\ &\Leftrightarrow \mu_{\underset{i\in J}{\cap} A_i}(x) \geq \alpha,\ \nu_{\underset{i\in J}{\cap} A_i}(x) \leq \beta \\ &\Leftrightarrow x \in (\bigcap_{i\in J} A_i)_{(\alpha,\beta)} \end{aligned}$$

This proves the conclusions (i) and (ii) of the theorem.

3.7. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFS in a nonempty set X; and $\{\alpha_i\}, \{\beta_i\}, i \in J$ be non empty subsets of $[0,1]$ with $\alpha_i + \beta_i \leq 1$. Let $\gamma_1 = \bigwedge_{i\in J} \alpha_i$, $\delta_1 = \bigvee_{i\in J} \beta_i$ and $\gamma_2 = \bigvee_{i\in J} \alpha_i$, $\delta_2 = \bigwedge_{i\in J} \beta_i$, then*

1. $\bigcup_{i\in J} A_{(\alpha_i,\beta_i)} \subseteq A_{(\gamma_1,\delta_1)}$

2. $\bigcap_{i\in J} A_{(\alpha_i,\beta_i)} = A_{(\gamma_2,\delta_2)}$

Proof.

$$\begin{aligned} \text{We have, } x \in \bigcup_{i\in J} A_{(\alpha_i,\beta_i)} &\Rightarrow x \in A_{(\alpha_j,\beta_j)}, \text{ forsome } j \in J \\ &\Rightarrow \mu_A(x) \geq \alpha_j, \nu_A(x) \leq \beta_j \\ &\Rightarrow \mu_A(x) \geq \alpha_j \geq \bigwedge_{i\in J} \alpha_i = \gamma_1, \nu_A(x) \leq \beta_j \leq \bigvee_{i\in J} \beta_i = \delta_1 \\ &\Rightarrow x \in A_{(\gamma_1,\delta_1)}. \end{aligned}$$

$$\begin{aligned} \text{Also, } x \in \bigcap_{i\in J} A_{(\alpha_i,\beta_i)} &\Leftrightarrow x \in A_{(\alpha_i,\beta_i)}, \ \forall\, i \in J \\ &\Leftrightarrow \mu_A(x) \geq \alpha_i, \nu_A(x) \leq \beta_i, \ \forall\, i \in J \\ &\Leftrightarrow \mu_A(x) \geq \bigvee_{i\in J} \alpha_i = \gamma_2, \nu_A(x) \leq \bigwedge_{i\in J} \beta_i = \delta_2 \\ &\Rightarrow x \in A_{(\gamma_2,\delta_2)}. \end{aligned}$$

From these two observations the theorem follows.

3.8. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFS in an R-module M. Then A is an IFSM of M if and only if for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$, $A_{(\alpha,\beta)}$ is a submodule of M.*

Proof. Suppose $A = (\mu_A, \nu_A)$ is an IFSM of an R-module M. Let $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$.

$$\begin{aligned} \text{We have, } x, y \in A_{(\alpha,\beta)} &\Rightarrow \mu_A(x) \geq \alpha, \mu_A(y) \geq \alpha \text{ and } \nu_A(x) \leq \beta, \nu_A(y) \leq \beta \\ &\Rightarrow \mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha, \\ &\qquad\qquad \text{and } \nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y) \leq \beta \\ &\Rightarrow x - y \in A_{(\alpha,\beta)}. \\ \text{Also, } r \in R, x \in A_{(\alpha,\beta)} &\Rightarrow \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta \\ &\Rightarrow \mu_A(rx) \geq \mu_A(x) \geq \alpha \text{ and } \nu_A(rx) \leq \nu_A(x) \leq \beta \\ &\Rightarrow rx \in A_{(\alpha,\beta)}. \end{aligned}$$

Hence $A_{(\alpha,\beta)}$ is a submodule of M.

Conversely suppose $A_{(\alpha,\beta)}$ is a submodule of M for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then, since $0 \in A_{(1,0)}$ we get $\mu_A(0) = 1$ and $\nu_A(0) = 0$.

Now let $x, y \in M$ be arbitrarily fixed. Let $\mu_A(x) \wedge \mu_A(y) = \alpha$ and $\nu_A(x) \vee \nu_A(y) = \beta$. Then obviously $\mu_A(x) \geq \alpha$, $\nu_A(x) \leq \beta$. This implies that $x \in A_{(\alpha,\beta)}$. Similarly we get $y \in A_{(\alpha,\beta)}$. Hence $x + y \in A_{(\alpha,\beta)}$, since $A_{(\alpha,\beta)}$ is a submodule of M. From this it follows that $\mu_A(x+y) \geq \alpha = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x+y) \leq \beta = \nu_A(x) \wedge \nu_A(y)\ \forall\, x, y \in M$.

Also we have $x \in A_{(\alpha_1,\beta_1)}$, where $\alpha_1 = \mu_A(x)$ and $\beta_1 = \nu_A(x)$. Therefore for any $r \in R$, we get $rx \in A_{(\alpha_1,\beta_1)}$. From this we get $\mu_A(rx) \geq \alpha_1 = \mu_A(x)$ and $\nu_A(rx) \leq \beta_1 = \nu_A(x)$. That is $\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)\ \forall r \in R$, and $\forall x \in M$. Thus $A = (\mu_A, \nu_A)$ is an IFSM of M.

4 Characterizations of IFSM's

In this section we derive some necessary and sufficient conditions under which an IFS in an R-module becomes an IFSM.

4.1. Definition ([9]). Let X be a non-empty set and let $a, b \in (0,1]$ with $a+b \leq 1$. We define an intuitionistic fuzzy point (IFP) $A = (\mu_{a_{\{y\}}}, \nu_{b_{\{y\}}})$ of X to be the IFS in X which is defined as,

$$\mu_{a_{\{y\}}}(x) = \begin{cases} a & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad \text{and} \quad \nu_{b_{\{y\}}}(x) = \begin{cases} b & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad \forall\, x \in X$$

4.2. Definition. Let X be a non-empty set. The intuitionistic fuzzy point $\hat{1}_{\{0\}}$ in X is defined as $\hat{1}_{\{0\}} = (\mu_{\hat{1}_{\{0\}}}, \nu_{\hat{1}_{\{0\}}})$ where

$$\mu_{\hat{1}_{\{0\}}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \nu_{\hat{1}_{\{0\}}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad \forall\, x \in X$$

4.3. Proposition. *If $A = (\mu_A, \nu_A)$ is an IFS in an R-module M, then $r\hat{1}_{\{0\}} = \hat{1}_{\{0\}}$ for any $r \in R$.*

Proof. We have $r\hat{1}_{\{0\}} = (\mu_{r\hat{1}_{\{0\}}}, \nu_{r\hat{1}_{\{0\}}})$. Then

$$\mu_{r\hat{1}_{\{0\}}}(x) = \vee\{\mu_{\hat{1}_{\{0\}}}(y) : y \in M, x = ry\} = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Therefore $\mu_{r\hat{1}_{\{0\}}}(x) = \mu_{\hat{1}_{\{0\}}}(x)\ \forall x \in M$. Similarly we have $\nu_{r\hat{1}_{\{0\}}}(x) = \nu_{\hat{1}_{\{0\}}}(x)\ \forall x \in M$. Hence $(\mu_{r\hat{1}_{\{0\}}}, \nu_{r\hat{1}_{\{0\}}}) = (\mu_{\hat{1}_{\{0\}}}, \nu_{\hat{1}_{\{0\}}})$. That is $r\hat{1}_{\{0\}} = \hat{1}_{\{0\}}$.

4.4. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFS in M. Then A is an IFSM of M if and only if A satisfies the following conditions.*

1. $\hat{1}_{\{0\}} \subseteq A$
2. $rA \subseteq A\ \ \forall\, r \in R$
3. $A + A \subseteq A$

Proof. Suppose $A = (\mu_A, \nu_A)$ is an IFSM of M. Now $\hat{1}_{\{0\}} = (\mu_{\hat{1}_{\{0\}}}, \nu_{\hat{1}_{\{0\}}})$ where

$$\mu_{\hat{1}_{\{0\}}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \nu_{\hat{1}_{\{0\}}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad \forall\, x \in M.$$

Then obviously $\mu_{\hat{1}_{\{0\}}}(x) \leq \mu_A(x)$ and $\nu_{\hat{1}_{\{0\}}}(x) \geq \nu_A(x)\ \forall\, x \in M$.
Hence $(\mu_{\hat{1}_{\{0\}}}, \nu_{\hat{1}_{\{0\}}}) \subseteq (\mu_A, \nu_A)$. That is $\hat{1}_{\{0\}} \subseteq A$.

Now for any $r \in R$, we have $rA = (\mu_{rA}, \nu_{rA})$. Then $\mu_{rA}(x) = \vee\{\mu_A(y) : y \in M, x = ry\} \leq \mu_A(x)\ \forall\ x \in M$, since $\mu_A(x) = \mu_A(ry) \geq \mu_A(y)$ Similarly $\nu_{rA}(x) \geq \nu_A(x)\ \forall\ x \in M$. Thus $(\mu_{rA}, \nu_{rA}) \subseteq (\mu_A, \nu_A)$. Hence $rA \subseteq A\ \ \forall\ r \in R$. Again $\mu_{A+A}(x) = \vee\{\mu_A(y) \wedge \mu_A(z) : y, z \in M, x = y + z\} \leq \mu_A(x)\ \forall\ x \in M$, since $\mu_A(x) = \mu_A(y+z) \geq \mu_A(y) \wedge \mu_A(z)$. Similarly $\nu_{A+A}(x) \geq \nu_A(x)\ \forall\ x \in M$. Thus $(\mu_{A+A}, \nu_{A+A}) \subseteq (\mu_A, \nu_A)$. Hence $A + A \subseteq A$. Thus if $A = (\mu_A, \nu_A)$ is an IFSM of M then A satisfies the three conditions.

Conversely, suppose $A = (\mu_A, \nu_A)$ satisfies the three conditions given. We prove that A is an IFSM of M. Since $\hat{1}_{\{0\}} \subseteq A$, we have $\mu_{\hat{1}_{\{0\}}}(x) \leq \mu_A(x)$ and $\nu_{\hat{1}_{\{0\}}}(x) \geq \nu_A(x)\ \forall\ x \in M$, which implies that $\mu_A(0) = 1$ and $\nu_A(0) = 0$. Now for $x, y \in M$, $\mu_A(x + y) \geq \mu_{A+A}(x + y)$, by condition (2), which is $\geq \mu_A(x) \wedge \mu_A(y)$ (since $\mu_{A+A}(x+y) = \vee\{\mu_A(z_1) \wedge \mu_A(z_2) : z_1, z_2 \in M, x+y = z_1 + z_2\} \geq \mu_A(x) \wedge \mu_A(y)$). Similarly we get $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)\ \ \forall\ x, y \in M$.

Also for $r \in R, x \in M$, $\mu_A(rx) \geq \mu_{rA}(rx)$ by condition (2), which is $\geq \mu_A(x)$ (since $\mu_{rA}(rx) = \vee\{\mu_A(y) : y \in M, rx = ry\} \geq \mu_A(x)$). Similarly we get $\nu_A(rx) \leq \nu_A(x)\ \ \forall\ r \in R, x \in M$. Hence $A = (\mu_A, \nu_A)$ is an IFSM of M.

4.5. Theorem. *Let $A = (\mu_A, \nu_A)$ be an IFS in an R-module M, then A is an IFSM of M if and only if A satisfies the following:*

1. $\hat{1}_{\{0\}} \subseteq A$

2. $rA + sA \subseteq A \quad \forall\ r, s \in R$

Proof. Proof follows from the above theorem.

4.6. Theorem. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSM's of M. Then $A + B = (\mu_{A+B}, \nu_{A+B})$ is an IFSM of M.*

Proof. We prove that $A + B$ satisfies the three conditions of the theorem 4.4. That is

(i) $\hat{1}_{\{0\}} \subseteq A+B$, (ii) $r(A+B) \subseteq A+B\ \forall\ r \in R$, (iii) $(A+B)+(A+B) \subseteq A + B$.

(i). We have $\hat{1}_{\{0\}} = (\mu_{\hat{1}_{\{0\}}}, \nu_{\hat{1}_{\{0\}}})$ and $A+B = (\mu_{A+B}, \nu_{A+B})$. Now for $x \in M$ by definition, $\mu_{A+B}(x) = \vee\{\mu_A(y) \wedge \mu_B(z) : y, z \in M, x = y+z\}$. So $\mu_{A+B}(0) = \vee\{\mu_A(y) \wedge \mu_B(z) : y, z \in M, 0 = y+z\} = \mu_A(0) \wedge \mu_B(0) = 1 = \mu_{\hat{1}_{\{0\}}}(0)$. Clearly $\mu_{\hat{1}_{\{0\}}}(x) \leq \mu_{A+B}(x)$ when $x \neq 0$. Thus $\mu_{\hat{1}_{\{0\}}}(x) \leq \mu_{A+B}(x)\ \forall\ x \in M$. Similarly we get $\nu_{\hat{1}_{\{0\}}}(x) \geq \nu_{A+B}(x)\ \forall\ x \in M$. This proves condition (i).

(ii). For $r \in R, x \in M$,

$$\begin{aligned}\mu_{r(A+B)}(x) &= \vee\{\mu_{A+B}(y) : y \in M, x = ry\} \\ &= \vee\{\vee\{\mu_A(z_1) \wedge \mu_B(z_2) : z_1, z_2 \in M, y = z_1 + z_2\} : y \in M, x = ry\} \\ &= \vee\{\mu_A(z_1) \wedge \mu_B(z_2) : z_1, z_2 \in M, x = rz_1 + rz_2\} \\ &\leq \vee\{\mu_A(rz_1) \wedge \mu_B(rz_2) : z_1, z_2 \in M, x = rz_1 + rz_2\} \\ &\leq \vee\{\mu_A(u) \wedge \mu_B(v) : u, v \in M, x = u + v\} \\ &= \mu_{A+B}(x).\end{aligned}$$

Similarly we can obtain $\nu_{r(A+B)}(x) \geq \nu_{A+B}(x)\ \forall r \in R, x \in M$. Thus $r(A + B) \subseteq A + B,\ \forall\, r \in R$ proving the condition (ii).

(iii). Now for $x \in M$,

$$\begin{aligned}\mu_{(A+B)+(A+B)}(x) &= \vee\{\mu_{A+B}(y) \wedge \mu_{A+B}(z) : y, z \in M, x = y + z\} \\ &= \vee\{(\vee\{\mu_A(y_1) \wedge \mu_B(y_2) : y_1, y_2 \in M, y = y_1 + y_2\}) \\ &\qquad \wedge (\vee\{\mu_A(z_1) \wedge \mu_B(z_2) : z_1, z_2 \in M, z = z_1 + z_2\}) \\ &\qquad : y, z \in M, x = y + z\} \\ &= \vee\{(\mu_A(y_1) \wedge \mu_B(y_2)) \wedge (\mu_A(z_1) \wedge \mu_B(z_2)) \\ &\qquad : y_1, y_2, z_1, z_2 \in M, x = y_1 + y_2 + z_1 + z_2\} \\ &= \vee\{(\mu_A(y_1) \wedge \mu_A(z_1)) \wedge (\mu_B(y_2) \wedge \mu_B(z_2)) \\ &\qquad : y_1, z_1, y_2, z_2 \in M, x = y_1 + z_1 + y_2 + z_2\} \\ &\leq \vee\{\mu_A(y_1 + z_1) \wedge \mu_B(y_2 + z_2) \\ &\qquad : y_1, z_1, y_2, z_2 \in M, x = y_1 + z_1 + y_2 + z_2\} \\ &\leq \vee\{\mu_A(u) \wedge \mu_B(v) : u, v \in M, x = u + v\} \\ &= \mu_{A+B}(x).\end{aligned}$$

Similarly we obtain $\nu_{(A+B)+(A+B)}(x) \geq \nu_{(A+B)}(x)\ \forall x \in M$. Thus $(A + B) + (A + B) \subseteq A + B$. This proves condition (iii). Therefore $A + B$ is an IFSM of M.

Now we can generalize the above theorem to an arbitrary family of IFSM's of M. For this purpose we need the following definition.

4.7. Definition. Let $A_i = (\mu_{A_i}, \nu_{A_i})$ $(i \in J, |J| > 1)$, be a family of IFSM's of M. Then $\sum_{i\in J} A_i = \{(x, \mu_{\sum\limits_{i\in J} A_i}(x), \nu_{\sum\limits_{i\in J} A_i}(x) : x \in M\}$, where,

$$\begin{aligned}\mu_{\sum\limits_{i\in J} A_i}(x) &= \vee\{\bigwedge_{i\in j} \mu_{A_i}(x_i) : x_i \in M, i \in J, \sum_{i\in J} x_i = x\}\ \forall x \in M, \\ \text{and}\quad \nu_{\sum\limits_{i\in J} A_i}(x) &= \wedge\{\bigvee_{i\in j} \nu_{A_i}(x_i) : x_i \in M, i \in J, \sum_{i\in J} x_i = x\}\ \forall x \in M,\end{aligned}$$

where, in $\sum\limits_{i\in J} x_i$, at most finitely many x_i's are not equal to zero. $\sum_{i\in J} A_i$ is called the weak sum of the A_i's.

4.8. Theorem. *Let $A_i = (\mu_{A_i}, \nu_{A_i})$ $(i \in J, |J| > 1)$, be a family of IFSM's of an R-module M. Then $\sum_{i\in J} A_i$ is an IFSM of M.*

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