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∞-Tuples of Bounded Linear Operators on Banach Space

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Abstract

In this paper, we introduce for an ∞-tuple of bounded linear operators on a Banach space and some conditions to an ∞-tuple to satisfying the Hypercyclicity Criterion.

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1 Introduction

Let $\mathcal{X}$ be a Banach space and $T_1, T_2, ...$ are commutative bounded linear operators on $\mathcal{X}$. By an ∞-tuple we mean the ∞-component $\mathcal{T} = (T_1, T_2, ...)$. For the ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ the set

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}... : k_i \geq 0, i = 1, 2, ...\}$$

is the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$, take $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. In other hand

$$Orb(\mathcal{T}, x) = \{{T_1}^{k_1}{T_2}^{k_2}...(x) : k_i \geq 0, i = 1, 2, ...\}.$$

The set $Orb(\mathcal{T}, x)$ is called, orbit of vector x under $\mathcal{T}$ and ∞-Tuple $\mathcal{T} = (T_1, T_2, ...)$ is called hypercyclic ∞-tuple, if there is a vector $x \in \mathcal{X}$ such that, the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{{T_1}^{k_1}{T_2}^{k_2}...(x) : k_i \geq 0, i = 1, 2, ...\}} = \mathcal{X}.$$

In this case, the vector x is called a hypercyclic vector for the ∞-tuple $\mathcal{T}$. All of operators in this paper are commutative bounded linear operators on a Banach space. Also, note that by $\{j,i\}$ or (j,i) we mean a number, that was showed with this mark, related with this indexes, not a pair of numbers. Readers can see $[1-10]$ for some information.

2 Preliminary Notes

Let $\mathcal{X}$ is a Banach space and $x \in X$, the vector x is called a semi-periodic vector for ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ if the sequence $\{T_1^{m_1}T_2^{m_2}...(x)\}$ be semi-compact, in this case $\mathcal{T}$ is called semi-periodic ∞-tuple. The vector x in $\mathcal{X}$ is called a Periodic vector for the ∞-Tuple $\mathcal{T} = (T_1, T_2, ...)$, if there exist some numbers $\mu_1, \mu_2, ... \in \mathcal{N}$ such that $T_1^{\mu_1}T_2^{\mu_2}...(x) = x$. The ∞-Tuple $\mathcal{T} = (T_1, T_2, ...)$, is called chaotic ∞-tuple, if we have tree below conditions together,
(1). It is topologically transitive, that is, if for any given open sets $\mathcal{U}$ and $\mathcal{V}$, there exist positive integer numbers $\alpha_1, \alpha_2, ... \in \mathcal{N}$ such that

$$T_1^{\alpha_1}T_2^{\alpha_2}...(\mathcal{U}) \cap \mathcal{V} \neq \phi$$

(2). It has a dense set of periodic points, in other word, there is a set $\mathcal{K}$ such that for each $x \in \mathcal{K}$, there exist some numbers $\beta_1, \beta_2, ... \in \mathcal{N}$ such that

$$T_1^{\beta_1}T_2^{\beta_2}...(x) = x$$

(3). It has a certain property called sensitive dependence on initial conditions. An ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is called syndetically hypercyclic if for any syndetic sequences of positive integers $\{m_{k,1}\}_k, \{m_{k,2}\}_k, ...$ the sequence $\{T_1^{m_{k,1}}T_2^{m_{k,2}}...\}_k$ is hypercyclic, in other hand, there is $x \in \mathcal{X}$, such that the set $\{T_1^{m_{k,1}}T_2^{m_{k,2}}...x : k \geq 0\}$ is dens in $\mathcal{X}$, that is, $\overline{\{T_1^{m_{k,1}}T_2^{m_{k,2}}...(x)\}} = \mathcal{X}$. The ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is called topologically mixing if for any given open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{X}$, there exist positive numbers K_1, K_2,... such that

$$T_1{}^{k_{1,i}}T_2{}^{k_{2,i}}...(\mathcal{U}) \cap \mathcal{V} \neq \phi \quad , \quad \forall k_{j,i} \geq K_j \quad , \quad \forall j = 1, 2, ...$$

The space $\mathcal{F}$ is called topologically mixing if for any given open sets $\mathcal{U}$ and $\mathcal{V}$, there exist positive numbers $M_1, M_2, ...$ such that

$$T_1^{m_{i,1}}T_2^{m_{i,2}}...(\mathcal{U}) \cap \mathcal{V} \neq \phi \quad , \quad \forall m_{i,j} \geq M_j \quad , \quad i = 1, 2, ...$$

The dense generalized kernel of ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$, is the set

$$\bigcup_{k>0}^{\infty}\bigcup_{i=1}^{\infty}(Ker(T_1^{i_{(1,k)}}T_2^{i_{(2,k)}}...)).$$

3 Main Results

If the ∞-tuple satisfying the hypothesis of the theorem 3.1, then we say that ∞-tuple satisfying the hypothesis of The Hypercyclicity Criterion.

Theorem 3.1 (The Hypercyclicity Criterion for ∞-Tuples) *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ...)$ is an ∞-tuple of continuous linear mappings on $\mathcal{X}$. If there exist two dense subsets $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{X}$, and strictly increasing sequences $\{m_{j,1}\}_{j=1}^{\infty}$, $\{m_{j,2}\}_{j=1}^{\infty}$, ... such that :*
1. $T_1^{m_{j,1}} T_2^{m_{j,2}} ... \to 0$ on $\mathcal{Y}$ as $j \to \infty$,
2. There exist functions $\{S_j : \mathcal{Z} \to \mathcal{X}\}$ such that for every $z \in \mathcal{Z}$, $S_j z \to 0$, and $T_1^{m_{j,1}} T_2^{m_{j,2}} ... S_j z \to z$, on $\mathcal{Z}$ as $j \to \infty$,
then $\mathcal{T}$ is a hypercyclic ∞-tuple.

Theorem 3.2 *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ...)$ is an hypercyclic ∞-tuple of commutative continuous linear mappings on $\mathcal{X}$. the ∞-tuple $\mathcal{T}$ satisfying the hypothesis of The Hypercyclicity Criterion, if there is a subset $\mathcal{S}$ of $\mathcal{X}$ such that, all elements of $\mathcal{S}$ are semi-periodic vectors for ∞-tuple $\mathcal{T}$.*

proof. Since $\mathcal{T}$ is hypercyclic ∞-tuple, then we can choice a vector $x \in \mathcal{X}$ such that $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$. Take $U_k = (0, \frac{1}{k})$ and $V_k = \{x + u : u \in U_k\}$. Since the set of hypercyclic vectors for any ∞-tuple is dense in $\mathcal{X}$, so let $u_1 \in U_1$ and the natural numbers $m_{1,1}, m_{2,1}, ...$ with property $T_1^{m_{1,1}} T_2^{m_{2,1}} ... \in V_1$. Now, take $u_2 \in U_2$. Since

$$\overline{\{T_1^{m_{1,1}+1} T_2^{m_{2,1}+1} ...(u_2), T_1^{m_{1,1}+2} T_2^{m_{2,1}+2} ...(u_2), ...\}} = \mathcal{X}$$

so there are $m_{1,2}, m_{2,2}, ...$ such that $T_1^{m_{1,2}} T_2^{m_{2,2}} ...(u_2) \in V_2$. Similarly, there are $m_{1,k}, m_{2,k}, ...$ such that $T_1^{m_{1,k}} T_2^{m_{2,k}} ...(u_k) \in V_k$. There are sequence $\{u_t\}_{k=1}^{\infty}$ of hypercyclic vectors and subsequences $\{m_{1,t}\}, \{m_{2,t}\}, ...$ of natural numbers, such that $lim_{k\to\infty}(u_k) = 0$ and $T_1^{m_{1,t}} T_2^{m_{2,t}} ...(u_t) \in V_t$. Now we try to find subsequence $\{m'_{1,k}\}, \{m'_{2,k}\}, ...$ of $\{m_{1,k}\}, \{m_{2,k}\}, ...$ such that

$$T_1^{m'_{1,k}} T_2^{m'_{2,k}} ...(u_2)(V_k) \bigcap (U_k) \neq \phi.$$

For this, suppose that $V = V_k$ and $U = U_k$ for any given k. If $\mathcal{S}$ be the set of all semi-periodic vectors of $\mathcal{T}$ then $\overline{\mathcal{S}} = \mathcal{X}$ so $\mathcal{S} \bigcap (x + B(0, \frac{1}{2k})) \neq \phi$, in other word, we can take $\omega \in \mathcal{S} \bigcap (x + B(0, \frac{1}{2k}))$ indeed the orbit of $\{T_1^{m_{1,k}} T_2^{m_{2,k}} ...(u_k)\}$ is semi-compact, so there are subsequences $\{\eta_{1,j}\}, \{\eta_{2,j}\}, ...$ of $\{m_{1,j}\}, \{m_{2,j}\}, ...$ such that $\{T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ...(\omega)\}$ is a convergence sequence. Suppose $\omega_0 \in \mathcal{X}$ and $T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ...(\omega) \to \omega_0$ as $j \to \infty$. With replace $\epsilon = \frac{1}{2k}$ we have k_0 and

$$\|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ...(\omega) - \omega_0\| \leq \frac{1}{2k_0}$$

as $\eta_j > k$. since x be a hypercyclic vector for $\mathcal{S}$ then there are natural numbers $\{\eta_{1,t}\}, \{\eta_{2,t}\}, ...$ such that,

$$\|T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(\omega) + \omega\| \leq \frac{1}{2k}.$$

Since $lim(u_i) = 0$ as $i \to 0$, then there is r_α such that

$$\| u_\alpha \| \leq \frac{1}{2k_0 \| T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}... \|}.$$

For k, take η_i with property $\eta_i > k$ and $\eta_i > k_0$, now we have

$$\|T_1^{\eta_{1,k}}T_2^{\eta_{2,k}}...T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(u_i) + \omega\| \leq \frac{1}{k}.$$

Since

$$\|\omega + T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(u_i) - x\| = \|\omega - x + T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(u_i)\| \leq \frac{1}{k}$$

then $h = \omega + T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(u_i) \in V_k$, so clearly

$$\|T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(h)\| = \|T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(\omega + T_1^{\eta_{1,t}}T_2^{\eta_{2,t}}...(u_i))\| \leq \frac{1}{k}$$

that is

$$T_1^{\eta_{1,k}}T_2^{\eta_{2,k}}...(V_k) \bigcap U_k \neq \phi$$

By this the proof is complete.

Theorem 3.3 *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ...)$ is an ∞-tuple of commutative bounded linear mapping on $\mathcal{X}$. If $\mathcal{T}$ is a hypercyclic ∞-tuple and it have a dense generalized kernel, then ∞-tuple $\mathcal{T}$ satisfying the hypothesis of The Hypercyclicity Criterion.*

proof. Since $\mathcal{T}$ is a hypercyclic ∞-tuple, then take the hypercyclic vector x for $\mathcal{T}$. So the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{{T_1}^{k_1}{T_2}^{k_2}...(x) : k_i \geq 0, i = 1, 2, ...\}} = \mathcal{X}.$$

Suppose that $\mathcal{F}$ be the generalized kernel of $\mathcal{T}$ and

$$\mathcal{N} = \{x, T_1^{m_{1,k}}T_2^{m_{2,k}}...(x), T_1^{m_{1,k}+1}T_2^{m_{2,k}+1}...(x)), ...\}.$$

Since $\overline{\mathcal{M}} = \mathcal{X}$ and x is a hypercyclic vector for $\mathcal{T}$, then

$$\overline{\mathcal{N}} = \mathcal{X} \tag{1}$$

Now we can take increasing sequences of positive integers $\{\kappa_{1,j}\}, \{\kappa_{2,j}\}, ...\}$ and $\{\nu_j\}$ such that $\nu_j \to 0$ and $T_1{}^{\kappa_{1,j}} T_2{}^{\kappa_{2,j}}...(\nu_j) \to x$ as $j \to \infty$. Since $T_1{}^{\kappa_{1,j}} T_2{}^{\kappa_{2,j}}...(\nu_j)$ is a hypercyclic vector for the ∞-tuple $\mathcal{T}$, so

$$T_1{}^{\kappa_{1,j}+1} T_2{}^{\kappa_{2,j}+1}...(\nu_j)$$

is a hypercyclic vector. By this, we choice the sequences $\{\eta_{1,j}\}, \{\eta_{2,j}\}, ...$, such that $T_1{}^{\eta_{1,j}} T_2{}^{\eta_{2,j}}...(x) \to x$ and $\{\kappa_{1,t}\} > \{\kappa_{1,j}\}, \{\kappa_{2,t}\} > \{\kappa_{2,j}\}, ... > \{\kappa_{n,j}\}$ as $j \to \infty$. So, there are $\{\mu_{1,j}\}, \{\mu_{2,j}\}, ...$ such that

$$\kappa_{1,j} + \mu_{1,j} = \eta_{1,j}, \kappa_{2,j} + \mu_{2,j} = \eta_{2,j}, ...$$

So $T_1{}^{\eta_{1,j}} T_2{}^{\eta_{2,j}}...(\omega_j) \to x$ as $j \to \infty$. Now by (1) defined $\mathcal{S}_{n_k} : \mathcal{N} \Rightarrow\to \mathcal{X}$ by $\mathcal{S}_{n_k}(T_1{}^{\eta_{1,j}} T_2{}^{\eta_{2,j}}...(x)) = T_1{}^{\eta_{1,j}} T_2{}^{\eta_{2,j}}...(\omega_k)$ we have $\eta_{1,j} = 0, 1, 2, ...$ as $j = 1, 2, ...$ Now we have $\mathcal{M}$ and $\mathcal{N}$ and $\mathcal{S}_{n_k} : \mathcal{N} \to \mathcal{X}$ that satisfying the hypothesis of Hypercyclicity Criterion for the ∞-tuple $\mathcal{T}$.

Theorem 3.4 *An ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is hereditarily hypercyclic with respect to increasing sequences of non-negative integers $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$, if and only if for all given any two open sets $\mathcal{U}$, $\mathcal{V}$, there exist some positive integers $M_1, M_2, ...$ such that $T_1^{m_{k,1}} T_2^{m_{k,2}}...(\mathcal{U}) \bigcap \mathcal{V} \neq \phi$ for $\forall m_{k,1} > M_1, \forall m_{k,2} > M_2, ...$*

Proof. Let $\mathcal{T} = (T_1, T_2, ...)$ be hereditarily hypercyclic ∞-tuple with respect to increasing sequences of non-negative integers $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$, and suppose that there exist some open sets $\mathcal{U}$, $\mathcal{V}$ such that $T_1^{m_{k,1}} T_2^{m_{k,2}}...(\mathcal{U}) \bigcap \mathcal{V} = \phi$ for some subsequence $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, ...$ of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$ respectively. Since the ∞-tuple $\mathcal{T} = (T_1, T_2, ...)$ is hereditarily hypercyclic with respect to $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$, thus $\{T_1^{m'_{k,1}} T_2^{m'_{k,2}}...\}$ is hypercyclic, and so we get a contradiction.

Conversely, suppose that $\{m'_{j,1}\}_{j=1}^{\infty}, \{m'_{j,2}\}_{j=1}^{\infty}, ...$ are arbitrary subsequences of $\{m_{j,1}\}_{j=1}^{\infty}, \{m_{j,2}\}_{j=1}^{\infty}, ...$ respectively, and $\mathcal{U}$, $\mathcal{V}$ are open sets in $\mathcal{X}$, satisfying

$$T_1^{m_{k,1}} T_2^{m_{k,2}}...(\mathcal{U}) \bigcap \mathcal{V} \neq \phi$$

for any $m_{(k,j)} > M_j, j = 1, 2, ...$ So there exist (i, j), large enough for $j = 1, 2, ...$ such that $m_{(k_i,j)} > M_j$ for $j = 1, 2, ...$ and

$$T_1^{m_{(k_i,1)}} T_2^{m_{(k_i,2)}}...(\mathcal{U}) \bigcap \mathcal{V} \neq \phi.$$

This implies that $\{T_1^{m_{(k_i,1)}} T_2^{m_{(k_i,2)}}...\}$ is hypercyclic, so the ∞-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is indeed hereditarily hypercyclic with respect to the sequences $\{m_{(k,1)}\}_{k=1}^{\infty}, \{m_{(k,2)}\}_{k=1}^{\infty}, ...$ By this the proof is complete.

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