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Do Linear Transformations Preserve Fuzzy Linear Independence? Some Examples

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Abstract

In this paper, we examine whether basic linear transformations in $\mathbb{R}^2$, such as rotations, scales, etc, preserve fuzzy linear independence. We provide some examples to the contrary.

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1. Introduction

It is a well-known fact in linear algebra, that invertible linear transformations preserve linear independence of vectors (see [2]). One would like to examine whether this is also true for fuzzy linearly independent vectors. As the examples in the next section show, this is actually *not* the case. But first, some preliminary definitions:

Lubczonok in [1] defines a fuzzy vector space to be a pair (V, μ), where V is a vector space and μ is a membership function $\mu : V \longrightarrow [0, 1]$ that satisfies:

$$\mu(ax + by) \geq \mu(x) \wedge \mu(y)$$

where $x, y \in V, a, b \in \mathbb{R}$.

Also in [1], we have that given a fuzzy vector space (V, μ), a finite set of vectors $\{x_i\}_{i=1}^n$ in V are *fuzzy linearly independent* if and only if $\{x_i\}_{i=1}^n$ are linearly independent and for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$ we have:

$$\mu(\sum_{i=1}^n a_i x_i) = \bigwedge_{i=1}^n \mu(a_i x_i).$$

2. Linear Transformations and Fuzzy Linear Independence - Some Examples

In this section, we present examples which show that basic invertible linear transformations (which preserve classical linear independence) do not in general preserve fuzzy linear independence. This shows that extra conditions are probably required, in addition to linearity, invertibility, etc, in order to preserve fuzzy linear independence. We examine four types of linear transformations: rotations, reflections, shears and scales:

Example 1: Let $(\mathbb{R}^2, \mu)$, where $\mu : V \longrightarrow [0,1]$ is defined as follows:

$$\mu \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 1\ , & x = 0,\ y = 0 \\ 1\ , & x = 0,\ y \neq 0 \\ 0\ , & x \neq 0,\ y = 0 \\ 0\ , & x \neq 0,\ y \neq 0. \end{cases}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Consider the vectors $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which are clearly linearly independent, as well as fuzzy linearly independent as one can easily check. Rotating these vectors by $45°$ counterclockwise, under $R_{45°} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$, we get $x' = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ and $y' = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$. These vectors are obviously still linearly independent, but not fuzzy linearly independent. Indeed, for $a = b \neq 0$, we have:

$$\mu(ax' + by') = \mu(a \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} + a \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ a\sqrt{2} \end{pmatrix} = 1 \neq 0 = 0 \wedge 0 = \mu(ax') \wedge \mu(by')$$

which shows that these vectors are *not* fuzzy linearly independent.

Example 2: Let $(\mathbb{R}^2, \mu)$, where $\mu : V \longrightarrow [0,1]$ and the vectors x and y are all defined as in Example 1. Again, these vectors are clearly linearly independent, as well as fuzzy linearly independent, but reflecting these vectors about the y-axis under $L_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we get $x' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $y' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These vectors are obviously still linearly independent, but not fuzzy linearly independent, because for $a = 0$, $b \neq 0$, we have:

$$\mu(ax' + by') = \mu(0 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ b \end{pmatrix} = 1 \neq 0 = 0 \wedge 1 = \mu(ax') \wedge \mu(by')$$

which shows that these vectors are *not* fuzzy linearly independent.

Example 3: Let $(\mathbb{R}^2, \mu)$, where $\mu : V \longrightarrow [0,1]$ and the vectors x and y are all defined as in Example 1. Again, these vectors are clearly linearly independent, as well as fuzzy linearly independent. Applying a shear $H_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \neq 0$, to these vectors, we get $x' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y' = \begin{pmatrix} k \\ 1 \end{pmatrix}$. These vectors are obviously still linearly independent, but not fuzzy linearly independent, because for $a \neq 0$, $b = -a/k$, we have:

$$\mu(ax' + by') = \mu(a \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{a}{k} \begin{pmatrix} k \\ 1 \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ -a/k \end{pmatrix} = 1 \neq 0 = 0 \wedge 0 = \mu(ax') \wedge \mu(by')$$

which shows that these vectors are *not* fuzzy linearly independent.

Example 4: Let $(\mathbb{R}^2, \mu)$, where $\mu : V \longrightarrow [0,1]$ and the vectors x and y are all defined as in Example 1. Once again, these vectors are clearly linearly independent, as well as fuzzy linearly independent. Applying a scale $S_{c,d} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, $c, d > 0$, to these vectors, we get $x' = \begin{pmatrix} c \\ 0 \end{pmatrix}$ and $y' = \begin{pmatrix} 0 \\ d \end{pmatrix}$. These vectors are obviously still linearly independent, but not fuzzy linearly independent, because for $a = 0$, $b \neq 0$, we have:

$$\mu(ax' + by') = \mu(0 \begin{pmatrix} c \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ d \end{pmatrix}) = \mu \begin{pmatrix} 0 \\ bd \end{pmatrix} = 1 \neq 0 = 0 \wedge 1 = \mu(ax') \wedge \mu(by')$$

which shows that these vectors are *not* fuzzy linearly independent.

Conclusion

As we showed above, basic invertible linear transformations such as rotations, reflections, shears and scales do *not* preserve fuzzy linear independence. Therefore, considering the fact that in general invertible linear transformations are combinations of some of the basic transformations mentioned above, then one concludes that in general linear transformations do *not* preserve fuzzy linear independence. An interesting future project could be to find those extra (necessary and/or sufficient) conditions (if any) that would guarantee fuzzy linear independence under linear transformations.

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References

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[2] L. Smith, Linear Algebra, Springer, 3rd ed., Ch.8, 1998.

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