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Solution of Nonlinear Differential Equations Using Mixture of Elzaki Transform and Differential Transform Method

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Abstract

In this paper, we apply a new integral transform "Elzaki transform" with the differential transform method to solve some nonlinear differential equations .The method is based on the ELzaki transform and differential transform methods. The nonlinear terms can be easily handled by the use of differential transform method.

Keywords: ELzaki transform – differential transform- nonlinear differential equations.

Introduction

Many problems of physical interest are described by ordinary or partial differential equations with appropriate initial or boundary conditions, these problems are usually formulated as initial value Problems or boundary value problems, ELzaki transform method [1-4] is particularly useful for finding solutions for these problems.
ELzaki transform is a useful technique for solving linear Differential equations but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. This paper is using differential transforms method [5, 6, 7,] to decompose the nonlinear term, so that the solution can be obtained by iteration procedure. This means that we can use both ELzaki transform and differential transform methods to solve many nonlinear Problems. The main aim of this paper is to solve nonlinear differential equations by using of ELzaki differential transform method. The main thrust of this technique is that the solution which is expressed as an infinite series converges fast to exact solutions.

ELzaki Transform:

Consider functions in the set A defined by:

$$A=\left\{f(t):\exists M,k_1,k_2>0,\ |f(t)|<Me^{|t|/k_j},\qquad if\quad t\in(-1)^j\times[0,\infty)\right\}$$

Where M a constant must be a finite number and k_1,k_2 can be finite or infinite.
ELzaki transform denoted by the operator $E(\cdot)$, is defined by the integral equation:

$$E[f(t)]=T(u)=u\int_0^\infty f(t)e^{-\frac{t}{u}}dt\ ,\ k_1\le u\le k_2,t\ge 0 \tag{1}$$

Theorem (1): [1]

Let $T(u)$ be ELzaki transform of $f(t)$ $[E(f(t))=T(u)]$ then:

(i) $E[f'(t)]=\frac{T(u)}{u}-uf(0)$ (ii) $E[f''(t)]=\frac{T(u)}{u^2}-f(0)-uf'(0)$

Proof:-

(i) By the definition we have:

$E[f'(t)]=u\int_0^\infty f'(t)e^{-\frac{t}{u}}dt$, Integrating by parts, we get:

$$E[f'(t)]=\frac{T(u)}{u}-uf(0)$$

(ii) Let $g(t)=f'(t)$, Then: $E[g'(t)]=\frac{1}{u}E[g(t)]-ug(0)$

Using (i) to find that: $E[f''(t)]=\frac{T(u)}{u^2}-f(0)-uf'(0)$

Differential Transform:

Differential transform of the function $y(x)$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0} \tag{2}$$

And the inverse differential transform of $Y(k)$ is defined as:

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^k$$

The main theorems of the one – dimensional differential transform are.

Theorem (2) If $w(x)=y(x)\pm z(x)$, then $W(k)=Y(k)\pm Z(x)$

Theorem (3) If $w(x)=cy(x)$, Then $W(k)=cY(k)$

Theorem (4) If $w(x)=\frac{dy(x)}{dx}$, then $W(k)=(k+1)Y(k+1)$

Theorem (5) If $w(x)=\frac{d^n y(x)}{dx^n}$, then $W(k)=\frac{(k+n)!}{k!}Y(k+n)$

Theorem (6) If $w(x)=y(x)z(x)$, then $W(x)=\sum_{r=0}^{k} Y(r)Z(k-r)$

Theorem (7) If $w(x)=x^n$, then $W(k)=\delta(k-n)=\begin{cases}1 & , \quad k=n\\ 0 & , \quad k\neq n\end{cases}$

Note that c is a constant and n is a nonnegative integer.

Analysis of Differential Transform:

In this section, we will introduce a reliable and efficient algorithm to calculate the differential transform of nonlinear functions.

I / Exponential nonlinearity: $f(y)=e^{ay}$.

From the definition of transform

$$F(0)=\left[e^{ay(x)}\right]_{x=0}=e^{ay(0)}=e^{aY(0)} \tag{3}$$

By differentiation $f(y)=e^{ay}$ with respect to x, we get:

$$\frac{df(y)}{dx}=ae^{ay}\frac{dy(x)}{dx}=af(y)\frac{dy(x)}{dx} \tag{4}$$

Application of the differential transform to Eq (4) gives:

$$(k+1)F(k+1)=a\sum_{m=0}^{k}(m+1)Y(m+1)F(k-m) \tag{5}$$

Replacing $k+1$ by k gives

$$F(k)=a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-m),\ k\geq 1 \tag{6}$$

Then from Eqs (3) and (6), we obtain the recursive relation

$$F(k)=\begin{cases} e^{aY(0)} & , \quad k=0\\ a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)\,F(m-1-m) & , \quad k\geq 1\end{cases} \tag{7}$$

II / Logarithmic nonlinearity: $f(y)=\ln(a+by)$, $a+by>0$.

Differentiating $f(y)=\ln(a+by)$, with respect to x, we get:

$$\frac{df(y(x))}{dx}=\frac{b}{a+by}\frac{dy(x)}{dx}, \text{ or } a\frac{df(y)}{dx}=b\left[\frac{dy(x)}{dx}-y\frac{df(y)}{dx}\right] \tag{8}$$

By the definition of transform:

$$F(0)=\left[\ln(a+by(x))\right]_{x=0}=\ln\left[a+by(0)\right]=\ln\left[a+bY(0)\right] \tag{9}$$

Take the differential transform of Eq.(8) to get:

$$aF(k+1)=b\left[Y(k+1)-\sum_{m=0}^{k}\frac{m+1}{k+1}F(m+1)Y(k-m)\right] \tag{10}$$

Replacing $k+1$ by k yields:

$$aF(k)=b\left[Y(k)-\sum_{m=0}^{k-1}\frac{m+1}{k}F(m+1)Y(k-1-m)\right],\ k\geq 1 \tag{11}$$

Put $k=1$ into Eq.(11) to get:

$$F(1)=\frac{b}{a+by(0)}Y(1). \tag{12}$$

For $k\geq 2$, Eq. (11) can be rewritten as

$$F(k)=\frac{b}{a+by(0)}\left[Y(k)-\sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1)Y(k-1-m)\right] \tag{13}$$

Thus the recursive relation is:

$$F(k)=\begin{cases}\ln\left[a+bY(0)\right] & ,\quad k=0\\ \dfrac{b}{a+bY(0)}Y(1) & ,\quad k=1\\ \dfrac{b}{a+bY(0)}\left[Y(k)-\sum_{m=0}^{k-2}\dfrac{m+1}{k}F(m+1)Y(k-1-m)\right] & ,\quad k\geq 2\end{cases}$$

Application:

Example (1)

Consider the simple nonlinear first order differential equation.

$$y'=y^2 \quad , \qquad y(0)=1 \tag{14}$$

First applying ELzaki transform (E) on both sides to find:

$$\frac{Y(u)}{u}-uy(0)=E\left[y^2\right]\Rightarrow Y(u)=u^2+uE\left[y^2\right] \tag{15}$$

The standard ELzaki transformation method defines the solution $y(t)$ by the series.

$$y=\sum_{n=0}^{\infty}y(n) \tag{16}$$

Operating with ELzaki inverse on both sides of Eq (15) gives:

$$y(t)=1+E^{-1}\left[u\ E\left(y^2\right)\right] \tag{17}$$

Substituting Eq (16) into Eq (17) we find:

$$y(n+1)=E^{-1}\left\{u\ E\left[A_n\right]\right\} \ ,\ n\geq 0 \tag{18}$$

Where $\quad A_n=\sum_{r=0}^{n}y(r)\,y(n-r)$, and $A_0=1$

For $n=0$, we have: $\quad y(1)=E^{-1}\{u\ E\ [A_0]\}=E^{-1}\{u\ E\ [1]\}=t$

For $n=1$, we have: $\quad A_1=2t$ and $y(2)=E^{-1}\{u\ E[2t]\}=t^2$

For $n=2$,we haves: $\quad A_2=3t^2$ and $y(3)=E^{-1}\{u\ E[3t^2]\}=t^3$

The Solution in a series form is given by.

$$y(t)=y(0)+y(1)+y(2)+y(3)+......\Rightarrow y(t)=1+t+t^2+t^3+.........=\frac{1}{1-t}$$

Example (2)

We consider the following nonlinear differential equation.

$$\frac{dy}{dt}=y-y^2 \quad , \quad y(0)=2 \tag{19}$$

In a similar way we have:

$$\frac{Y(u)}{u}-uy(0)=E\left[y-y^2\right] \text{ or } Y(u)=2u^2+uE\left[y-y^2\right] \tag{20}$$

The inverse of ELzaki transform implies that:

$$y(t)=2+E^{-1}\left\{u\,E\left[y-y^2\right]\right\} \tag{21}$$

The recursive relation is given by:

$$y(n+1)=E^{-1}\left\{u\,E\,\left[y(n)-A_n\right]\right\},\ n\geq 0 \tag{22}$$

Where $y(0)=2$, and $A_n=\sum_{r=0}^{n} y(r)y(n-r)$

The first few components of A_n are

$$A_0=y^2(0)\ ,\ A_1=2y(0)y(1)\ ,\ A_2=2y(0)y(2)+y^2(1)$$
$$A_3=2y(0)y(3)+2y(1)y(2)\ ,\ ......$$

From the recursive relation we have:

$$y(0)=2$$
$$y(1)=E^{-1}\left\{u\,E\left[y(0)-A_0\right]\right\}=E^{-1}\left\{u\,E\left[-2\right]\right\}=-2t$$
$$y(2)=E^{-1}\left\{uE\left[y(1)-A_1\right]\right\}=E^{-1}\left\{uE\left[6t\right]\right\}=3t^2$$
$$y(3)=E^{-1}\left\{u\,E\left[y(2)-A_2\right]\right\}=E^{-1}\left\{u\,E\left[-13t^2\right]\right\}=-\frac{13}{3}t^3$$

Then we have the following approximate solution to the initial problem.

$$y(t)=y(0)+y(1)+y(2)+....$$
$$y(t)=2-2t+3t^2-\frac{13}{3}t^3+\frac{25}{4}t^4......=\frac{2}{2-e^{-t}}$$

Example (3)

Consider the nonlinear initial – value Problem

$$y''(x)=2y+4y\ln y\ ,\ y>0\ ,\ y(0)=1\ ,\ y'(0)=0 \tag{23}$$

Applying ELzaki transform to Eq (23) and using the initial conditions, we obtain.

$$Y(u)=u^2+u^2E\left[2y+4y\ln y\right] \tag{24}$$

Take the inverse of Eq (24) to find:

$$y(x)=1+E^{-1}\left\{u^2E\left[2y+4y\ln y\right]\right\} \tag{25}$$

The recursive relation is given by:

$$y(n+1) = E^{-1}\left\{u^2 E\left[2y(n) + 4A_n\right]\right\} \tag{26}$$

Where $A_n = \sum_{m=0}^{n} y(m) F(n-m)$ (27)

And $F(n) = \begin{cases} \ln\left(y(0)\right) & , \quad n = 0 \\ \dfrac{y(1)}{y(0)} & , \quad n = 1 \\ \dfrac{y(n)}{y(0)} - \sum_{m=0}^{n-2} \dfrac{m+1}{ny(0)} F(m+1) y(n-1-m) & , \quad n \geq 2 \end{cases}$ (28)

Then we have:

$F(0) = 0 \Rightarrow A_0 = 0$, and $y(1) = E^{-1}\left\{u^2 E[2]\right\} = E^{-1}\left[2u^4\right] = x^2$

$F(1) = x^2 \Rightarrow A_1 = x^2$, and $y(2) = E^{-1}\left\{u^2 E\left[6x^2\right]\right\} = \dfrac{x^4}{2}$

$F(2) = 0 \Rightarrow A_2 = x^4$, and $y(3) = E^{-1}\left\{u^2 E\left[5x^4\right]\right\} = \dfrac{x^6}{6}$

Then the exact solution is:

$$y(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + ... = \sum_{k=0}^{\infty} \frac{1}{k!}\left(x^2\right)^k = e^{x^2}$$

Example (4)

Consider the initial –value problem of Bratu-type.

$$y''(x) - 2e^y = 0 \quad , \quad 0 < x < 1 \quad , \quad y(0) = y'(0) = 0 \tag{29}$$

Take ELzaki transform of this equation and use the initial condition to obtain:

$$Y(u) = u^2 E\left[2e^y\right] \tag{30}$$

Take the inverse to obtain:

$$y(x) = E^{-1}\left\{u^2 E\left[2e^y\right]\right\}$$

Then the recursive relation is given by:

$$y(n+1) = E^{-1}\left\{u^2 E\left[2F(n)\right]\right\} \tag{31}$$

Where $y(0) = 0$, and $F(n) = \begin{cases} e^{y(0)} & , \quad n = 0 \\ \sum_{m=0}^{n-1} \dfrac{m+1}{n} y(m+1) F(n-m-1) & , \quad n \geq 1 \end{cases}$ (32)

Then from Eqs (31) and (32) we have

$$F(0)=1 \text{ , and } \quad y(1)=E^{-1}\{u^2E[2]\}=E^{-1}[2u^4]=x^2$$

$$F(1)=x^2, \text{ and } \quad y(2)=E^{-1}\{u^2E[2x^2]\}=\frac{x^4}{6}$$

$$F(2)=\frac{2}{3}x^4, \text{ and } \quad y(3)=E^{-1}\left\{u^2E\left[\frac{4}{3}x^4\right]\right\}=\frac{2}{45}x^6$$

Then the series solution is

$$y(x)=x^2+\frac{1}{6}x^4+\frac{2}{45}x^6+......=-2\ln(\cos x)$$

Conclusions:

In this paper, the series solutions of nonlinear differential equations are obtained by using ELzaki transform and differential transform methods. This technique is useful to solve linear and nonlinear differential equations.

Appendix:

Elzaki transform of some Functions

$f(t)$	$E[f(t)]=T(u)$
1	u^2
t	u^3
t^n	$n!\ u^{n+2}$
e^{at}	$\frac{u^2}{1-au}$
$sinat$	$\frac{au^3}{1+a^2u^2}$
$cosat$	$\frac{u^2}{1+a^2u^2}$

References

[1] S. lslam, Yasir Khan, Naeem Faraz and Francis Austin, Numerical Solution of Logistic Differential Equations by using the Laplace Decomposition Method, World Applied Sciences Journal 8 (9) :1100-1105,2010.

[2] Nuran Guzel and Muhammet Nurulay,Solution of Shiff Systems By using Differential Transform Method, Dunlupinar universities Fen Bilimleri Enstitusu Dergisi, 2008 ISSN 1302-3055, PP. 49-59.

[3] Shin- Hsiang Chang , I-Ling Chang, A new algorithm for calculating one – dimensional differential transform of nonlinear functions, Applied Mathematics and Computation 195 (2008) 799-808.
[4] Tarig M. Elzaki, The New Integral Transform "Elzaki Transform" Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 57-64.
[5] Tarig M. Elzaki & Salih M. Elzaki, Application of New Transform "Elzaki Transform" to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 65-70.
[6] Tarig M. Elzaki & Salih M. Elzaki, On the Connections Between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 6, Number 1(2011),pp. 1-11.
[7] Tarig M. Elzaki & Salih M. Elzaki, On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6, Number 1(2011),pp. 13-18.

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