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Exact Peakon Solutions of a Generalized CH-KP Equation[1]

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Abstract

In this paper, the peakons of a generalized CH-KP equation are studied by using bifurcation and simulation methods. The representations of peakons are given, and their planar graphs are showed. These results are supplement to investigate CH-KP equation.

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1 Introduction and main results

Wazwaz [1] considered the following water wave equations given by

$$[u_t + 2ku_x - u_{xxt} - au^n u_x)_x + u_{yy} = 0, \qquad (1.1)$$

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and

$$[u_t + 2ku_x - u_{xxt} + au^n(u^n)_x]_x + u_{yy} = 0, \tag{1.2}$$

where $a > 0$, $k \in R$ and n is called the strength of the nonlinearity. Eqs. (1.1) and (1.2) are called CH - KP equations. Lai et al. [2] studied the generalized forms of CH - KP, which are written by

$$[u_t + 2ku_x - (u^m)_{xxt} - au^n u_x]_x + u_{yy} = 0, \tag{1.3}$$

and

$$[u_t + 2ku_x - (u^m)_{xxt} + au^n(u^n)_x]_x + u_{yy} = 0. \tag{1.4}$$

and derived families of exact travelling wave solutions of Eqs. (1.3) and (1.4). Biswas [3] obtained an exact 1 - soliton solution of (1.3) and (1.4) by the solitary wave ansatze. Zhang et al. [4] showed that Eq. (1.3) has some smooth and non-smooth travelling wave solutions by using the bifurcation theory of planar dynamical systems, and also obtained the expressions of solitons, compactons and periodic solutions, under some especial conditions. However, it is usually difficult to solve gCH - KP for arbitrary m or n, so we study the peakon solutions of following gCH - KP.

$$[u_t + 2ku_x - (u^2)_{xxt} - auu_x]_x + u_{yy} = 0. \tag{1.5}$$

In this paper, by using bifurcation and simulation methods [5-7], the peakons of Eq. (1.5) are studied for $a < 0$. The explicit expressions of peakons are given.

In order to state our main results conveniently, for given constant $a < 0, c \neq 0$ and $g \neq 0$, let $g_1 = \frac{(2k-c+1)^2}{2a}$, $g_2 = \frac{4(2k-c+1)^2}{9a}$ and $\varphi_\pm = \frac{2k-c+1 \pm \sqrt{(2k-c+1)^2-2ag}}{a}$ for $g > g_1(c)$.

Proposition 1. *If $k < -\frac{1}{2}, c < 2k+1, g = g_2$ or $k \geq \frac{1}{2}, c < 0, g = g_2$, then the Eq. (1.5) has a peakon solution (see Fig. 1 (a)) as follows:*

$$u(x, y, t) = \varphi_+[1 - \exp(-\sqrt{\frac{a}{8c}}|x + y - ct|)]. \tag{1.6}$$

Proposition 2. *If $k < -\frac{1}{2}, 2k+1 < c < 0, g = g_2$, then Eq. (1.5) has a peakon solution (see Fig. 1 (b)) as follows:*

$$u(x,y,t) = \varphi_-[1 - \exp(-\sqrt{\frac{a}{8c}}|x+y-ct|)]. \tag{1.7}$$

Example 1. Letting $a = -1, k = -2, c = -5$, then $g_1 = -2$ and $g_2 = -\frac{16}{9}$. Taking $g = -\frac{16}{9}$, we have $\varphi_- = -\frac{4}{3}$ and $\varphi_+ = -\frac{8}{3}$. Substituting these data into (1.6), on $\xi - u$ plane we draw a peakon graph as Fig.1 (a), where $\xi = x+y-ct$.

Example 2. Letting $a = -1, k = -4, c = -1$, then $g_1 = -18$ and $g_2 = -16$. Taking $g = -16$, we have $\varphi_- = 8$ and $\varphi_+ = 4$. Substituting these data into (1.7), on $\xi - u$ plane we draw a peakon graph as Fig.1 (b), where $\xi = x + y - ct$.

2 Preliminary

For $a < 0$, making the transformation $u(x,y,t) = \varphi(\xi)$ with $\xi = x+y-ct$ in Eq. (1.5), we have:

$$[(2k-c)\varphi' + c(\varphi^2)''' - a\varphi\varphi']' + \varphi'' = 0, \tag{2.1}$$

where c is the wave speed. We note that time t is lost as $c = 0$, so we just discuss the case $c \neq 0$.

Integrating (2.1) once with respect to ξ and neglecting integral constant, we have the following ordinary differential equation:

$$(2k - c + 1)\varphi' - a\varphi\varphi' + c(\varphi^2)''' = 0. \tag{2.2}$$

Integrating (2.2) once, we have the following travelling wave equation:

$$(2k - c + 1)\varphi - \frac{a}{2}\varphi^2 + c(\varphi^2)'' = g, \tag{2.3}$$

where g is integral constant. Zhang et al. [4] have studied the case $g = 0$, so we just discuss the case $g \neq 0$.

Let $\varphi' = z$, Eq. (2.3) becomes the following two dimensional system:

$$\begin{cases} \frac{d\varphi}{d\xi} = z \\ \frac{dz}{d\xi} = \frac{\frac{a}{2}\varphi^2 - (2k-c+1)\varphi + g - 2cz^2}{2c\varphi} \end{cases}, \tag{2.4}$$

which is called travelling wave system.

Let

$$d\xi = 2c\varphi d\tau, \tag{2.5}$$

then system (2.4) becomes:

$$\begin{cases} \frac{d\varphi}{d\tau} = 2c\varphi z \\ \frac{dz}{d\tau} = \frac{a}{2}\varphi^2 - (2k-c+1)\varphi + g - 2cz^2 \end{cases}. \tag{2.6}$$

Thus systems (2.4) and (2.6) have same fist integral

$$H(\varphi, z) = 2c\varphi^2 z^2 - \frac{a}{4}\varphi^4 + \frac{2(2k-c+1)}{3}\varphi^3 - g\varphi^2 = h. \tag{2.7}$$

Using the dynamical system theory of planar systems, we know that the singular points of system (2.6) have following properties.

(1) When $gc > 0$, $(0, \pm\sqrt{\frac{g}{2c}})$ are two saddle points.

(2) When $g = g_1$, $(\frac{2k-c+1}{a}, 0)$ is a degenerate saddle point.

(3) When $g > 0$ and $k \leq -\frac{1}{2}, c > 0$ or $k > -\frac{1}{2}, c > 0$, $(\varphi_-, 0)$ and $(\varphi_+, 0)$ are two center points.

(4) When $g > 0$ and $k \leq -\frac{1}{2}, c < 0$ or $k > -\frac{1}{2}, c < 0$, $(\varphi_-, 0)$ and $(\varphi_+, 0)$ are two saddle points.

(5) When $g_1 < g < 0, c < 0$ and $2k - c + 1 > 0$, $(\varphi_-, 0)$ is a center point, $(\varphi_+, 0)$ is a saddle point.

(6) When $g_1 < g < 0, c < 0$ and $2k - c + 1 < 0$, $(\varphi_-, 0)$ is a saddle point, $(\varphi_+, 0)$ is a center point.

(7) When $g_1 < g < 0, c > 0$ and $2k - c + 1 > 0$, $(\varphi_-, 0)$ is a saddle point, $(\varphi_+, 0)$ is a center point.

(8) When $g_1 < g < 0, c > 0$ and $2k - c + 1 < 0$, $(\varphi_-, 0)$ is a center point, $(\varphi_+, 0)$ is a saddle point.

According to the above analysis, we draw the some bifurcation phase portraits of (2.4) and (2.6) which are show in Fig. 2.

3 The proof of main results

Based on the above analysis, we compute the peakon wave solutions of Eq. (1.5). From (2.7), because a cusp wave correspond a non-smooth close orbit, therefore we can get a cusp wave solution integrating along a non-smooth close orbit.

Case 1 $k < -\frac{1}{2}, c < 2k+1, g = g_2$ or $k \geq \frac{1}{2}, c < 0, g = g_2$.

In this case, the orbit of system (2.6) which passes through $(\varphi_+, 0)$ is a non-smooth close orbit (see Fig. 2 (a) Γ_1), it has expression as follows:

$$z = \pm\sqrt{\frac{a}{8c}}(\varphi - \varphi_+), \tag{3.1}$$

where $\varphi_+ < 0$.

Substituting (3.1) into $\frac{d\varphi}{d\xi} = z$ and integrating it along Γ_1, we have

$$\int_\varphi^0 \frac{1}{s - \varphi_+} ds = \int_\xi^0 \sqrt{\frac{a}{8c}} ds, \quad \text{for} \quad \xi < 0, \tag{3.2}$$

and

$$\int_0^\varphi \frac{1}{s - \varphi_+} ds = \int_0^\xi \sqrt{\frac{a}{8c}} ds, \quad \text{for} \quad \xi > 0. \tag{3.3}$$

From (3.2) and (3.3), we obtain a peakon solution $u(x, y, t) = \varphi(\xi)$ as follows:

$$\varphi(\xi) = \varphi_+[1 - \exp(-\sqrt{\frac{a}{8c}}|\xi|)]. \tag{3.4}$$

Here we complete the proof of Proposition 1.

Case 2. $k < -\frac{1}{2}, 2k+1 < c < 0, g = g_2$.

In this case, the orbit of system (2.6) which passes through $(\varphi_-, 0)$ is a non-smooth close orbit (see Fig. 2 (b) Γ_2), it has expression as follows:

$$z = \pm\sqrt{\frac{a}{8c}}(\varphi_- - \varphi), \tag{3.5}$$

where $\varphi_- > 0$.

Substituting (3.5) into $\frac{d\varphi}{d\xi} = z$ and integrating it along Γ_2, we have

$$\int_0^\varphi \frac{1}{\varphi_- - s} ds = \int_0^\xi \sqrt{\frac{a}{8c}} ds, \quad \text{for} \quad \xi > 0, \tag{3.6}$$

and

$$\int_{\varphi}^{0} \frac{1}{\varphi_- - s} ds = \int_{\xi}^{0} \sqrt{\frac{a}{8c}} ds, \quad \text{for} \ \ \xi < 0. \tag{3.7}$$

From (3.6) and (3.7), we obtain a peakon solution $u(x, y, t) = \varphi(\xi)$ as follows:

$$\varphi(\xi) = \varphi_-[1 - \exp(-\sqrt{\frac{a}{8c}}|\xi|)]. \tag{3.8}$$

Here we complete the proof of Proposition 2.

4 Conclusion

In this paper, the bifurcation and global behavior a CH - KP equation have been studied, and the conditions that peakons appear, and their representations are obtained. Their planar graphs are simulated under the some parameters (see Fig. 1).

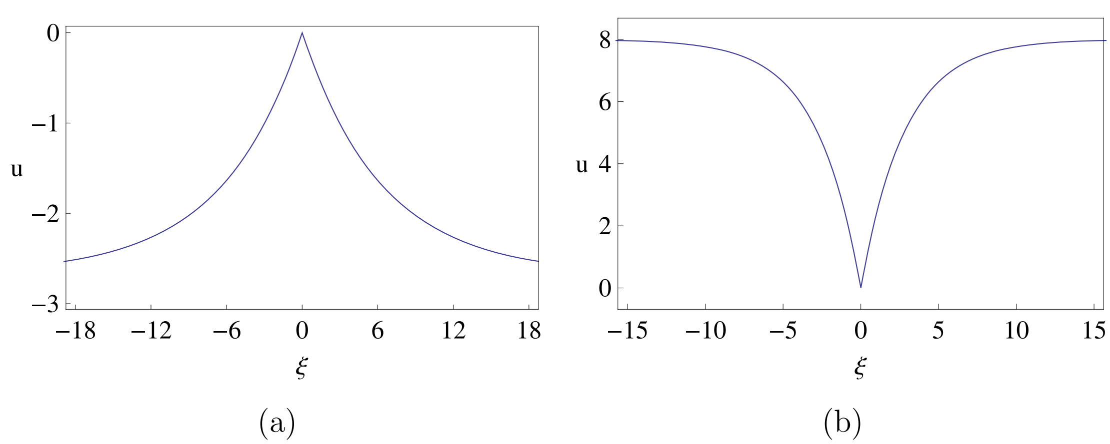
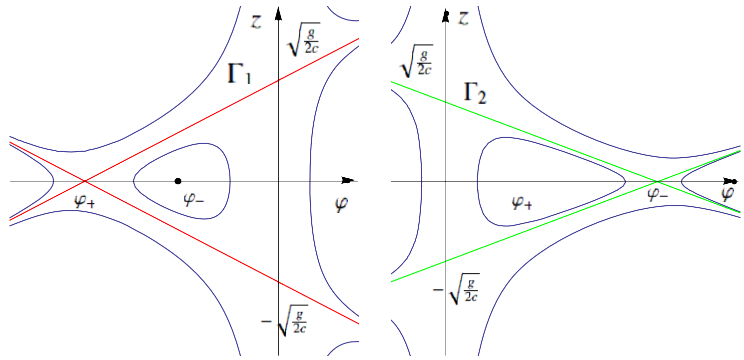


Fig. 1. The peakons of Eq.(1.5) with $a = -1$. (a) $k = -2, c = -5, g = -\frac{16}{9}$, (b) $k = -4, c = -1, g = -16$.



(a) (b)

Fig. 2. The bifurcation phase portraits of (2.4) and (2.6). (a) $k < -\frac{1}{2}, c < 2k+1$ and $g = g_2$ or $k \geq \frac{1}{2}, c < 0$ and $g = g_2$. (b) $k < -\frac{1}{2}, 2k+1 < c < 0$ and $g = g_2$.

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