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Rings Over which Every Module is Strongly n-Gorenstein Flat

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Abstract

This paper introduces and studies a particular subclass of the class of commutative rings with finite Gorenstein global dimension.

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1 Introduction

Throughout the paper, all rings are commutative with identity, and all modules are unitary.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finitely generated R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [7, 8] introduced the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda, and Torrecillas [10] introduced the Gorenstein flat dimension. Some references are [4, 5, 6, 7, 8, 10, 11].

Recall that an R-module M is called Gorenstein projective, if there exists an exact sequence of projective R-modules:

$$\mathbf{P} : ... \to P_1 \to P_0 \to P^0 \to P^1 \to ...$$

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such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that the functor $\mathrm{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective R-module. The complex $\mathbf{P}$ is called a complete projective resolution. The Gorenstein injective R-modules are defined dually.

An R-module M is called Gorenstein flat, if there exists an exact sequence of flat R-modules:

$$\mathbf{F} : ... \to F_1 \to F_0 \to F^0 \to F^1 \to ...$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that the functor $I \otimes_R -$ leaves $\mathbf{F}$ exact whenever I is an injective R-module. The complex $\mathbf{F}$ is called a complete flat resolution.

The Gorenstein projective, injective, and flat dimensions are defined in terms of resolutions and denoted by $\mathrm{Gpd}(-)$, $\mathrm{Gid}(-)$, and $\mathrm{Gfd}(-)$, respectively ([5, 9, 11]).

In [4], the authors proved, for any associative ring R, the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid \text{M is a (left) R-module}\} = \sup\{\mathrm{Gid}_R(M) \mid \text{M is a (left) R-module}\}.$$

They called the common value of the above quantities the left Gorenstein global dimension of R and denoted it by $l.\mathrm{Ggldim}(R)$. Similarly, they set

$$l.\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid \text{M is a (left) R-module}\}$$

which they called the left Gorenstein weak dimension of R. Since in this paper all rings are commutative, we drop the letter l.

Recently, in [13], particular modules of finite Gorenstein projective, injective, and flat dimensions are defined as follows:

Definitions 1.1. *Let n be a positive integer.*

1. *An R-module M is said to be strongly n-Gorenstein projective, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ where $\mathrm{pd}_R(P) \leq n$ and $\mathrm{Ext}_R^{n+1}(M, Q) = 0$ whenever Q is projective.*

2. *An R-module M is said to be strongly n-Gorenstein injective, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$ where $\mathrm{id}_R(I) \leq n$ and $\mathrm{Ext}_R^{n+1}(E, M) = 0$ whenever E is injective.*

3. *An R-module M is said to be strongly n-Gorenstein flat, if there exists a short exact sequence of R-modules $0 \longrightarrow M \longrightarrow F \longrightarrow M \to 0$ where $\mathrm{fd}_R(F) \leq n$ and $\mathrm{Tor}_R^{n+1}(M, I) = 0$ whenever I is injective.*

In [15], the authors defined and investigate the rings over which every module is strongly n-Gorenstein projective. In this paper, we complete the analogy. Hence, introduce and investigate the rings over which every module is strongly n-Gorenstein flat.

2 Main Results

Definition 2.1. *Let n be a positive integer. A ring R is called weakly n-strongly Gorenstein (n-wSG ring for short), if every R-module is strongly n-Gorenstein flat.*

The 0-wSG rings and 1-wSG rings are already studied in [12, 14] and they are called strongly Gorenstein Von Neumann regular rings, and strongly Gorenstein semihereditary rings (when the ring is coherent), respectively. Clearly, by definition, every n-wSG ring is m-wSG whenever $n \leq m$.

Our first result gives a characterization of weakly strongly n-Gorenstein rings.

Proposition 2.2. *Let R be a ring and n be a positive integer. The following conditions are equivalent.*

1. *R is an n-wSG ring.*
2. $\mathrm{wGgldim}(R) \leq n$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$
where $\mathrm{fd}_R(F) < \infty$.
3. $\mathrm{wGgldim}(R) < \infty$ *and for every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$
where $\mathrm{fd}_R(F) \leq n$.
4. *For every R-module M there exists a short exact sequence of R-modules*
$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$
where $\mathrm{fd}_R(F) \leq n$.

We need the following lemma.

Lemma 2.3 ([2], Theorem 2.2). *For an R-module M,* $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ *with equality if* $\mathrm{fd}_R(M) < \infty$.

Proof of Proposition 2.2. (1 ⇒ 2) Follows from the fact that every strongly n-Gorenstein flat module has a Gorenstein flat dimension $\leq n$ (by [13, Proposition 3.2(1)]).

(2 ⇒ 3) Let F be an R-module such that $\mathrm{fd}_R(F) < \infty$. Clearly, $\mathrm{Gfd}_R(F) \leq n$ since $\mathrm{wGgldim}(R) \leq n$. Then, by Lemma 2.3, $\mathrm{fd}_R(F) \leq n$.

(3 ⇒ 4) Obvious.

(4 ⇒ 1) Let I be an injective R-module. By hypothesis, there is an exact sequence of R-modules $0 \to I \to F \to I \to 0$ where $\mathrm{fd}_R(F) \leq n$. Clearly this exact sequence splits. Thus, $I \oplus I \cong F$. Hence, $\mathrm{fd}_R(I) \leq n$. Thus, for each R-module M, $\mathrm{Tor}_R^{n+1}(M, I) = 0$. Hence, each R-module is strongly n-Gorenstein flat. Consequently, R is an n-wSG ring.

Recall that a ring R is called perfect if every flat R-module is projective.

Theorem 2.4. *Every n-SG ring is n-wSG with equivalence in the following two cases:*

1. *R is Noetherian.*
2. *R is perfect with finite Gorenstein global dimension.*

We need the following lemma.

Lemma 2.5. *For any arbitrary ring R we have:* $\mathrm{wGgldim}(R) \leq \mathrm{Ggldim}(R)$ *with equality if R is Noetherian or perfect with finite Gorenstein global dimension.*

Proof. The desired inequality follows from [4, Corollary 1.2 (1)]. While, if R is Noetherian, the converse inequality follows from the equivalence [9, Theorem 12.3.1(3 ⇔ 4)].
If R is perfect with $n := \mathrm{Ggldim}(R) < \infty$, we claim that every Gorenstein flat R-module is Gorenstein projective. Let M be an arbitrary Gorenstein flat R-module. By the definition of Gorenstein flat modules, we can pick an n-step right flat resolution as follows:

$$0 \longrightarrow M \longrightarrow F_1 \longrightarrow F_1 \longrightarrow ... \longrightarrow F_n \longrightarrow G$$

where all F_i are flats and so projectives since R is perfect. But $\mathrm{Gpd}_R(G) \leq n$. Thus, using [11, Theorem 2.20($i \Leftrightarrow iv$)], we conclude that M is Gorenstein projective, as desired. Consequently, $\mathrm{wGgldim}(R) \leq \mathrm{Ggldim}(R)$ and this finishes the proof.

Proof of of Theorem 2.4. Using [15, Proposition 2.3] and Proposition 2.2, it is clear that every n-SG ring is an n-wSG ring. Now, if R is an n-wSG ring which is Noetherian or perfect with finite Gorenstein global dimension then $\mathrm{wGgldim}(R) = \mathrm{Ggldim}(R) \leq n$. Thus, Using [4, Corollary 2.7], we have $\mathrm{pd}_R(F) \leq n$ for each flat R-module with finite flat dimension. Thus, by [15, Proposition 2.3] and Proposition 2.2, R is an n-SG ring.

The next result studies the direct product of n-SG rings and n-wSG rings.

Theorem 2.6. *Let $\{R_i\}_{i=1}^m$ be a family of rings and set $R := \prod_{i=1}^{m} R_i$. Then, R is an n-wSG ring if and only if R_i is an n-wSG ring for each $i = 1, .., m$.*

Proof. By induction on m it suffices to prove the assertion for $m = 2$. First suppose that $R_1 \times R_2$ is an n-SG ring. We claim that R_1 is an n-wSG ring. Let M be an arbitrary R_1 module. $M \times 0$ can be viewed as an $R_1 \times R_2$-module. For such module and since $R_1 \times R_2$ is an n-wSG ring, there is an exact sequence $0 \longrightarrow M \times 0 \longrightarrow F \longrightarrow M \times 0 \longrightarrow 0$ where $\mathrm{pd}_{R_1 \times R_2}(F) \leq n$. Thus, since R_1 is a projective $R_1 \times R_2$ module, by applying $- \otimes_{R_1 \times R_2} R_1$ to the sequence above, we find the short exact sequence of R-modules: $0 \longrightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \longrightarrow F \otimes_{R_1 \times R_2} R_1 \longrightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \longrightarrow 0$. Clearly $\mathrm{fd}_{R_1}(P \otimes_{R_1 \times R_2} R_1) \leq \mathrm{pd}_{R_1 \times R_2}(P) \leq n$. Moreover, we have the isomorphism of R-modules:

$$M \times 0 \otimes_{R_1 \times R_2} R_1 \cong M \times 0 \otimes_{R_1 \times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M.$$

Thus, we obtain an exact sequence of R-module with the form: $0 \longrightarrow M \longrightarrow F \otimes_{R_1 \times R_2} R_1 \longrightarrow M \longrightarrow 0$. Thus, using Proposition 2.2, R_1 is an n-wSG ring, as desired. By the same argument, R_2 is also an n-wSG ring.
Now, suppose that R_1 and R_2 are an n-wSG rings and we claim that $R_1 \times R_2$ is an n-wSG ring. Let M be an $R_1 \times R_2$-module. We have

$$M \cong M \otimes_{R_1 \times R_2} (R_1 \times R_2) \cong M \otimes_{R_1 \times R_2} ((R_1 \times 0) \oplus (R_2 \times 0)) \cong M_1 \times M_2$$

where $M_i = M \otimes_{R_1 \times R_2} R_i$ for $i = 1, 2$. For each $i = 1, 2$, there is an exact sequence $0 \longrightarrow M_i \longrightarrow P_i \longrightarrow M_i \longrightarrow 0$ where $\mathrm{fd}_{R_i}(F_i) \leq n$ since R_i is an n-wSG ring. Thus, we have the exact sequence of $R_1 \times R_2$-modules:

$$0 \longrightarrow M_1 \times M_2 \longrightarrow F_1 \times F_2 \longrightarrow M_1 \times M_2 \longrightarrow 0.$$

On the other hand, $\mathrm{fd}_{R_1 \times R_2}(F_1 \times F_2) = \sup\{\mathrm{fd}_{R_i}(P_i)\}_{1,2} \leq n$ (by [3, Lemma 3.7]). Thus, from Proposition 2.2, $R_1 \times R_2$ is an n-wSG ring, as desired.

References

[1] M. Auslander and M. Bridger, Stable module theory, *Memoirs. Amer. Math. Soc.,* **94**, American Mathematical Society, Providence, R.I., 1969.

[2] D. Bennis, *A Note on Gorenstein Flat Dimension*, Algebra Colloquium 1(2011) pp. 155-161.

[3] D. Bennis and N. Mahdou; *Global Gorenstein dimensions of polynomial rings and of direct products of rings*, Houston Journal of Mathematics **25** (4), (2009), 1019-1028.

[4] D. Bennis and N. Mahdou; *Global Gorenstein Dimensions*, Proc. Amer. Math. Soc., **138** (2), (February 2010), 461-465.

[5] L. W. Christensen, *Gorenstein dimensions,* Lecture Notes in Math., 1747, Springer, Berlin, 2000.

[6] L. W. Christensen, A. Frankild, and H. Holm, On Gorenstein projective, injective and flat dimensions - a functorial description with applications, *J. Algebra* **302** (2006), 231-279.

[7] E. Enochs and O. Jenda, On Gorenstein injective modules, *Comm. Algebra,* **21** (10) (1993), 3489–3501.

[8] E. Enochs and O. Jenda, Gorenstein injective and projective modules, *Math. Z.,* **220** (4) (1995), 611–633.

[9] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Expositions in Mathematics, Walter de Gruyter and Co., Berlin, 2000.

[10] E. Enochs, O. Jenda and B. Torrecillas, Gorenstein flat modules, *Nanjing Daxue Xuebao Shuxue Bannian Kan,* **10** (1) (1993), 1–9.

[11] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra,* **189** (2004), 167-193.

[12] N. Mahdou and M. Tamekkante, On (strongly) Gorenstein (semi)hereditary rings, *Arab J Sci Eng,* (Springer) 36 (2011) 431–440.

[13] N. Mahdou and M. Tamekkante, Storongly n-Gorenstein projective, injective and flat modules, submitted for publication. Available from math.AC/0904.4013 v1 26 Apr 2009

[14] N. Mahdou, M. Tamekkante and S. Yassemi, On strongly Gorenstein Von Neumann regular rings, Communications in Algebra, accepted for publication.

[15] Mohamed Chhiti, Khalid Louartiti and Mohammed Tamekkante, On n-Strongly Gorenstein rings, Submitted for publication in "International Journal of Algebra".

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