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A Note on Joint Reduction of an Ideal over Local Ring

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Abstract. A criterion for the existence of joint reductions of the finite set of ideals over a local ring has been established in this paper.

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1. INTRODUCTION

Let A be a commutative Noetherian ring with identity having Krull dimension e and I be an ideal of A. Then ideal $J \subset I$ is said to be reduction of I if $JI^n = I^{n+1}$ for some non negative integer n. The least such integer n is called the reduction number of I relative to J. Reduction of an ideal [4], define a relationship between two ideals which are preserved under homomorphisms and ring extensions and it plays a role in the theory of finite- morphisms of the blow-up $Blow_{V(I)}(Spec(R))$.

The reduction number being a control element have a strong geometrical content, connection between reduction of an ideal and the homogeneous affine coordinate ring of the fiber over the closed point in the blow-up of the ring along the given ideal are explained in [5]. An ideal $J \subset I$ is a minimal reduction if it is not contained in any reduction of I. The minimal reductions have played an important role in the study of many problems in commutative algebra, for example Hilbert function, Rees algebras and associated graded rings. If A is a local ring of an infinite residue field, the minimal reductions are particularly valuable because they help to control the co-homology of the blow-up. Minimal reduction of an ideal have the pleasing property of carrying most of the information about the original ideal but in general with fewer generators. If the residue field k of ring A is infinite, then the minimal number of generators of a minimal reduction equals the analytic spread $l(I) := r$ of the ideal I, i.e; the dimension of the special fiber ring of I. Later, some authors (see [7], [10], [11], [12]) generalized the definition of reduction of an ideal to Joint reduction of ideal. Joint reduction of an ideal is a classical object that has attracted the attention of several researchers who have given treatments of it which has added to its understanding (see [5], [7], [8], [9], [11], [13]).

If (A, m, k) is a local ring of Krull dimension e with an infinite residue field $k = A/m$. Suppose $X_1, X_2, \ldots,$ is a countable set of indeterminates over A. Let A_g denotes the general extension of A, which is a localization of the ring $A[X_1, X_2, \ldots,]$ at the prime ideal $m[X_1, X_2, \ldots,]$. Rees and Sally ([8], Theorem 1.4) proved the following theorem for joint reduction. Let $(I_1, \ldots, I_e)$ be a set of e m-primary ideals of A, and $(x_1, \ldots, x_e)$, where $x_i \in I_i$ for $1 \leq i \leq e$ be an independent set of general elements of $(I_1, \ldots, I_e)$. Then $(x_1, \ldots, x_e)$ is a joint reduction of $(I_1A_g, \ldots, I_eA_g)$.

In this paper, we prove a variant of the above result without condition of an independent set of general elements using Boger's inequality ([3]) $l(I) := r \leq dim(A)$. We prove the following theorem in the Section 3:

Theorem 1.1. *Let (A, m) be a Noetherian local ring with infinite residue field k and I be an ideal of A having analytic spread r, where $I = I_1 \ldots . I_r$. Then there exist elements $x_i \in I_i$ for $i = 1, \ldots .r$ such that $(x_1, \ldots , x_r)$ is a joint reduction of $(I_1, \ldots, I_r)$.*

2. PRELIMINARIES

Definition 2.1. *The associated graded ring of I is defined as*

$$\mathfrak{R}_I(A) = \frac{A}{I} \oplus \frac{I}{I^2} \oplus \ldots.$$

If (A, m, k) is a local Noetherian ring with infinite residue field k. Then the reduction number of I can also be determined via the fiber cone

$$\mathfrak{F}_I(A) = \frac{A}{m} \oplus \frac{I}{mI} \oplus \ldots.$$

The analytic spread of I, denoted as $l(I)$, is defined to be Krull dimension of the special fiber ring $\mathfrak{F}_I(A)$. By Noetherian normalization lemma, there exist elements $a_1, \ldots , a_r \in I$ such that their images $b_1, \ldots , b_r \in I/mI$ are algebraically independent over k and $\mathfrak{F}_I(A)$ is an integral extension of $k[b_1, \ldots , b_r]$. It follows that there exists $n \geq 1$ so that $JI^n = I^{n+1}$, where $J = (a_1, \ldots , a_r)$. This implies that the smallest number of elements $a_1, \ldots , a_r$ required to generate a reduction of I is the analytic spread $l(I)$. Note that fiber cone $\mathfrak{F}_I(A)$ is the fiber over the closed point of the blow-up

$$Spec\left(\bigoplus_{n\geq 0} \frac{I^n}{I^{n+1}}\right) \to Spec(A).$$

and it plays an important role in the resolutions of the singularities of algebraic varieties (see [1]).

Definition 2.2. *Let $I_1, \dots, I_s$ be ideals of A and $a_i \in I_i$ for all $i = 1, \dots, s$. Then s-tuple $(a_1, \dots, a_s)$ is said to be a* joint reduction *of $(I_1, \dots, I_s)$ if the ideal $a_1 I_2 I_3 . \dots . I_s + a_2 I_1 I_3 . \dots . I_s + ... + a_s I_1 . \dots . I_{s-1}$ is a reduction of $I_1 . \dots . I_s$.*

In case $I = I_1 = \dots = I_s$, s-tuple $(a_1, \dots, a_s)$ being a joint reduction of $(I, \dots, I)$ says that $(a_1, \dots, a_s)I^{s-1}$ is a reduction of I^s, so that there exists an integer n such that $(a_1, \dots, a_s)I^{s-1}I^{sn} = I^{s(n+1)}$. In other words, s-tuple $(a_1, \dots, a_s)$ is a joint reduction of $(I, \dots, I)$ if and only if the ideal $(a_1, \dots, a_s)$ is a reduction of I. Thus joint reductions generalize the reductions.

Definition 2.3. *Let $\mathfrak{R}_I(A) = \bigoplus_{n \geq 0} \mathfrak{R}_I(A)_n$ be a Noetherian homogeneous ring. An element x of $\mathfrak{R}_I(A)$ is called a superficial element of $\mathfrak{R}_I(A)$ if $x \in \mathfrak{R}_I(A)_1$ and there is an element $d \geq 0$ such that*
$(0 :_{\mathfrak{R}_I(A)} x) \cap \mathfrak{R}_I(A)_n = 0$ for any $n \geq d$.

Theorem 2.4. [3](*Artin Rees*) *Let M be a finite module over a Noetherian ring A. Let N, N' be submodules of M and I be an ideal of A. Then there is a natural number d such that $I^n N \cap N' = I^{n-d}(I^d N \cap N')$ for any $n \geq d$.*

3. Results on the joint reduction of an ideal

Proposition 3.1. *Let (A, m, k) be a local ring with infinite residue field k and $I_1 . \dots . I_s$ be ideals in A, where $I = I_1 . \dots . I_s$. Suppose $V = I_1/mI_1 \otimes \dots \otimes I_s/mI_s$ be a finite dimensional vector space over an infinite field k and $W_1, \dots, W_t$ be proper subspaces of the k-vector space I/mI. Then there exist $x_i \in I_i$, $1 \leq i \leq s$, such that*

$$y + mI \notin W_1 \cup \dots \cup W_t,$$

where $y = x_1 . \dots . x_s$.

Proof. Proof is by induction on t. If $t = 1$, then clearly there exists $x_i \in I_i$ such that $y + mI \notin W_1$ for $1 \leq i \leq s$. If $t > 1$, then we can choose an element $\alpha + mI \in V$ such that $\alpha + mI \notin W_1 \cup \dots \cup W_{t-1}$ by inductive hypothesis. Since $W_t \subsetneq V$ there exists an element $\beta + mI \in V \setminus W_t$. Suppose that the result is false i.e. $W_1 \cup \dots \cup W_t = V$. Since k is infinite, there are distinct elements $r_1, \dots, r_{t+1}$ in k such that $\alpha + r_1\beta + mI, \dots, \alpha + r_{t+1}\beta + mI$ are in V. By the pigeon hole principle, two of them must be in the same subspace, say $\alpha + r_i\beta + mI$, $\alpha + r_j\beta + mI$ are in W_n for some n, where $i \neq j$. If $n = t$, then $(\alpha + r_i\beta + mI) - (\alpha + r_j\beta + mI) = (r_i - r_j)\beta + mI \in W_t$. Hence $\beta + mI \in W_t$, which is a contradiction to the choice of $\beta + mI$. If $n < t$, then $(r_i - r_j)\beta + mI \in W_n$, and hence $\beta + mI \in W_n$. Since $\alpha + r_i\beta + mI \in W_n$, it follows that $\alpha + mI \in W_n$, which is a contradiction to the choice of $\alpha + mI$. $\square$

Proposition 3.2. *Let (A, m) be Noetherian local ring and $I \subset m$ be ideal of A. Then $l(I) = 0$ iff I is nilpotent ideal. Thus if A has no non zero nilpotent elements, then $l(I) > 0$ for every non zero ideal I of A.*

Proof. Let $\Re := \frac{I}{mI} \oplus \frac{I^2}{mI^2} \oplus \dots$, denote the homogeneous maximal ideal of $\mathfrak{F}_I(A)$. Suppose $l(I) = 0$. Then $\Re$ is the unique homogeneous prime ideal of $\mathfrak{F}_I(A)$. Since $\mathfrak{F}_I(A)$ is graded ring, every minimal prime of $\mathfrak{F}_I(A)$ is homogeneous. Therefore $\Re$ is the unique prime ideal of $\mathfrak{F}_I(A)$. Hence $\Re = Rad(0)$ and $\Re^n = 0$ for some positive integer n. Thus $I^n = mI^n$ and by Nakayam's lemma $I^n = 0$. Then I is nilpotent ideal in A. Conversely let I be a nilpotent ideal. Then $\frac{I^n}{mI^n} \oplus \frac{I^{n+1}}{mI^{n+1}} \oplus \dots = 0$. Hence $\Re^n = 0$, so $\Re \subset P$ for each minimal primes P of $\mathfrak{F}_I(A)$. It follows that $\Re = P$ is the unique prime ideal of $\mathfrak{F}_I(A)$. Hence $l(I) = 0$. □

Theorem 3.3. *Let (A, m) be a Noetherian local ring with infinite residue field k and I be an ideal of A having analytic spread r, where $I = I_1 ... I_r$. Then there exist elements $x_i \in I_i$ for $i = 1, \dots .r$ such that $(x_1, \dots, x_r)$ is a joint reduction of $(I_1, \dots, I_r)$.*

Proof. The proof is by induction on r. By Proposition 3.2. $l(I) = 0$ if and only if I is a nilpotent ideal. So if $r = 0$, then $I^n = 0$ for $n \geq 1$. Therefore,

$$I^n = 0 = 0.I_1^{n-1} I_2^n . \dots .I_r^n + \dots + 0.I_1^n I_2^n . \dots .I_r^{n-1}.$$

Hence the result is true for $r = 0$.

Now, it suffices to consider the case where I contains a non zero divisor. The details are as follows: Set $K = (0 :_A I^q)$ for some $q \geq 1$ and consider the ring $\overline{A} = A/K$ and ideals $\overline{I_i} = \frac{I_i + K}{K}$ for all $i = 1, \dots, r$. If $\overline{I} = K$, i.e. $I + K = K$ and $I \subseteq K$, then $I^{q+1} = 0$ for $q \geq 1$. Thus I is a nilpotent ideal and $r = 0$. Therefore result is true in this case.

Now, if $\overline{I} \neq K$ i.e. $I \nsubseteq K$, then $(0 :_{\overline{A}} \overline{I}) = 0$. So $\overline{I}$ contains a non zero divisor. Since reductions are preserved under homomorphisms, and if $\overline{r} = l(\overline{I})$, then $\overline{r} \leq r$. Hence even if $\overline{r} = r$ and we suppose that the result is known when the ideal has a non zero divisor, or if $\overline{r} < r$ and by induction, we can find elements $x_i \in I_i$ for all $i = 1, \dots, \overline{r}$ such that

$$I^n = x_1 I_1^{n-1} I_2^n . \dots .I_{\overline{r}}^n + \dots + x_{\overline{r}} .I_1^n I_2^n . \dots .I_{\overline{r}}^{n-1} + K \cap I^n.$$

Multiplying both sides $(I_1 . \dots .I_r)^q$ we get,

$$I^{n+q} = x_1 I_1^{n+q-1} I_2^{n+q} . \dots .I_r^{n+q} + \dots + x_{\overline{r}} I_1^{n+q} I_2^{n+q} . \dots .I_r^{n+q-1}.$$

Now, we assume that $r \geq 1$ and I contains a non zero divisor. Consider the associated graded ring $\mathfrak{R}_I(A) = \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}}$ and fiber cone $\mathfrak{F}_I(A) = \bigoplus_{n \geq 0} \frac{I^n}{mI^n}$ of dimension r. Let $P_1, \dots, P_\mu$ be the associated prime ideal of

A and $Q_1, \ldots, Q_v$ be associated prime ideals of $\mathfrak{R}_I(A)$ which do not contain $\mathfrak{R}_I(A)_1$. Let $J_1, \ldots, J_w$ be the homogeneous primes in $\mathfrak{F}_I(A)$ of co-rank r. Set $t = \mu + v + w$, and set

$$W_i = (P_i + mI)/mI,\ 1 \leq i \leq \mu;$$

$$W_{\mu+i} = (Q_i \cap \mathfrak{R}_I(A)_1)/m\mathfrak{R}_I(A)_1,\ 1 \leq i \leq v;$$

$$W_{\mu+v+i} = J_i \cap \mathfrak{F}_I(A)_i,\ 1 \leq i \leq w;$$

Since I contains a non-zero divisor, W_i is a proper k-subspace of $\mathfrak{F}_I(A)_1$ for $i = 1, \ldots, \mu$ by Nakayama's lemma; similarly for $i = 1, \ldots, v$ and $W_{\mu+v+i}$ for $1 \leq i \leq w$ is a proper k-subspace of $\mathfrak{F}_I(A)_1$.
Therefore by Proposition 3.1, there exist $x_i \in I_i$ for all $i = 1, \ldots, r$ such that

1. $(y + mI)\mathfrak{F}_I(A)$ is of co-rank strictly less than r in $\mathfrak{F}_I(A)$, where $y = x_1. \ldots .x_r$ and each $x_i \in I_i$ for all $i = 1, \ldots, r$.
2. y is a non-zero divisor in A.
3. $y + I^2$ is a superficial element of $\mathfrak{R}_I(A)$ i.e. there exists a positive integer c such that
$$(I^n : yA) \cap I^c = I^{n-1}, \text{ when } n > c.$$
4. By Theorem 2.4 there exists a positive integer d such that $(I^n : yA) \subseteq I^{n-d}$, $n > d$. Since y is a non-zero divisor in A, it follows that $(I^n : yA) = I^{n-1}$ and $I^n \cap yA = yI^{n-1}$ for $n \geq c + d$.

Now write $\mathfrak{R}'_{I'}(A') = \dfrac{\mathfrak{R}_{I'}(A')}{y\mathfrak{R}_{I'}(A')}$ and $\mathfrak{F}_{I'}(A') = \bigoplus_{n\geq 0} \dfrac{I'^n}{m'I'^n}$, $I' = I/yA$, $A' = A/yA$ and $m' = m/yA$. Let $l(I') = r'$. Then $\mathfrak{F}_{I'}(A')$ has dimension $r' < r$. It follows by induction hypothesis applied to $\mathfrak{F}_{I'}(A')$ that there exist elements x_{ij}, $j = 1, \ldots, r'$ of I_i for $i = 1, \ldots, r$ such that for large n we get, $I^n = (y_1, \ldots, y_{r'})I^{n-1} + (yA \cap I^n)$, where $y_j = x_{1j}. \ldots .x_{rj}$ for $j = 1, \ldots, r'$. By using (4) we have,

$$I^n = (y_1, \ldots, y_r)I^{n-1} \subseteq x_1 I_1^{n-1} I_2^n. \ldots .I_r^n + \ldots + x_r I_1^n. \ldots .I_r^{n-1},$$

where $x_i = x_{i.j(i)}$, $i = 1, \ldots, r$ and $i \to j(i)$ is a permutation map of the set $\{1, \ldots, r\}$. Since $x_i \in I_i$, $x_i I_i^{n-1} \subseteq I_i^n$ for all $i = 1, \ldots, r$,

$$x_1 I_1^{n-1} I_2^n. \ldots .I_r^n + \ldots + x_r I_1^n. \ldots .I_r^{n-1} \subseteq I^n.$$

Hence equality holds

$$x_1 I_1^{n-1} I_2^n. \ldots .I_r^n + \ldots + x_r I_1^n. \ldots .I_r^{n-1} = I^n.$$

Thus $(x_1, \ldots, x_r)$ is a joint reduction of $(I_1, \ldots, I_r)$. □

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