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Recurrence Relations for Higher Moments of Order Statistics from Doubly Truncated Exponential Distribution

Narinder Pushkarna

Department of Statistics
Ramjas College, University of Delhi
Delhi-110007, India
drpushkarna@hotmail.com

Jagdish Saran and Rashmi Tiwari

Department of Statistics
University of Delhi, Delhi-110007, India
jagdish_saran52@yahoo.co.in
rashmistats@gmail.com

Abstract

In this paper, we have obtained higher moments of order statistics from doubly truncated exponential distribution which generalize the work done by Joshi (1978) and Balakrishnan and Joshi (1984).

Keywords : Order Statistics, Single moments, Product moments, Higher moments, Recurrence relations, Exponential distribution, Doubly truncated exponential distribution

1 INTRODUCTION

Let the random variable X have a doubly truncated exponential distribution with probability density function (pdf)

$$f(x) = \frac{e^{-x}}{P-Q}, \quad -log(1-Q) \leq x \leq -log(1-P) \tag{1.1}$$

and cumulative distribution function (cdf)

$$F(x) = \begin{cases} 0 & ; for\ x < Q_1 \\ \frac{1-Q-e^{-x}}{P-Q} & ; for\ Q_1 \leq x \leq P_1 \\ 1 & ; x > P_1 \end{cases} \tag{1.2}$$

where $Q_1 = -log(1-Q)$, $P_1 = -log(1-P)$ and Q and $1-P$ are the proportions of truncation on the left and right of the standard exponential distribution. The proportion Q and P, with $Q < P$, are assumed to be known.

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ represent an ordered sample of size n from the doubly truncated exponential distribution given in (1.1) and $X_{r_1:n} \leq X_{r_2:n} \leq \ldots \leq X_{r_k:n}$ be the corresponding sub-sample order statistics. Let us denote the single moments $E(X_{r:n}^k)$ by $\mu_{r:n}^{(k)}(1 \leq r \leq n)$ and the product moments of k order statistics, viz. $E(X_{r_1:n}^{i_1} X_{r_2:n}^{i_2} \ldots X_{r_k:n}^{i_k})$ by $\mu_{r_1,r_2,\ldots,r_k:n}^{(i_1,i_2,\ldots,i_k)}(1 \leq r_1 < r_2 < \ldots < r_k \leq n\ and\ i_1, i_2, \ldots, i_k = 0, 1, 2 \ldots .)$.
Joshi (1978, 1982) derived some recurrence relations for single and product moments of order statistics from standard exponential distribution and also for right truncated exponential distribution. Balakrishnan and Gupta (1992) extended these results for standard exponential distribution as well as for right truncated exponential distribution and derived relations that will enable one to find the moments and cross moments (of order up to 4) of order statistics. Saleh, Scott and Junkins (1975) derived exact (but somewhat cumbersome) explicit expressions for the first two single moments and the product moments of order statistics. Joshi (1979) and Balakrishnan and Joshi (1984) derived several recurrence relations satisfied by the single and the product moments of order statistics from doubly truncated exponential distribution. Khan, Yaqub and Parvez (1983) tabulated these quantities for some value of P, Q, and n from doubly truncated exponential distribution.

From (1.1) and (1.2) we observe that the characterizing differential equations for the doubly truncated exponential distribution are

$$f(x) = Q_2 - F(x) \tag{1.3}$$

and

$$f(x) = P_2 + [1 - F(x)], \tag{1.4}$$

where $Q_2 = (1-Q)/(P-Q)$ and $P_2 = (1-P)/(P-Q)$. In this paper, we shall use equations (1.3) and (1.4) to establish several recurrence relations satisfied by higher moments of order statistics from doubly truncated exponential distribution defined in (1.1), thus generalizing the work of Joshi (1978, 1982, 1979) and Balakrishnan and Joshi (1984) for standard exponential and truncated exponential distributions.

2 RECURRENCE RELATIONS

The joint density function of $X_{r_1:n}, X_{r_2:n}, \ldots,$ and $X_{r_k:n} (1 \le r_1 < r_2 < \ldots < r_k \le n)$ is given by

$$\begin{aligned} f_{r_1,r_2,\ldots,r_k:n}(x_1,\ldots,x_k) &= C_{r_1,r_2,\ldots,r_k:n} \ldots [F(x_1)]^{r_1-1}[F(x_2)-F(x_1)]^{r_2-r_1-1} \ldots \\ &\quad .[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-1}[1-F(x_k)]^{n-r_k} f(x_1)f(x_2)\ldots f(x_k), \\ &\qquad Q_1 \le x_1 < \ldots < x_k \le P_1, \end{aligned} \tag{2.1}$$

where

$$C_{r_1,r_2,\ldots,r_k:n} = \frac{n!}{(r_1-1)!(r_2-r_1-1)!\ldots(r_k-r_{k-1}-1)!(n-r_k)!}$$

(cf. David and Nagaraja (2003), p.12), and $f(x)$ and $F(x)$ are as given in equations (1.1) and (1.2). Then by making use of the characterizing differential equations in (1.3) and (1.4), we establish recurrence relations for the product moments of k order statistics.

Theorem 2.1. *For* $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ *and* $i_1, i_2, \ldots, i_k \geq 0$,

$$\mu^{(i_1+1,i_2,\ldots,i_k)}_{1,2,r_3,\ldots,r_k:n} = (i_1+1)\mu^{(i_1,i_2,\ldots,i_k)}_{1,2,r_3,\ldots,r_k:n} - nQ_2\Big\{\mu^{(i_1+i_2+1,r_3,\ldots,i_k)}_{1,r_3-1,\ldots,r_k-1:n-1} - Q_1^{i_1+1}\mu^{(i_2,i_3,\ldots,i_k)}_{1,r_3-1,\ldots,r_k-1:n-1}\Big\} + \mu^{(i_1+i_2+1,\ldots,i_k)}_{2,r_3,\ldots,r_k:n} \qquad (2.2)$$

and for $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ *and* $i_1, i_2, \ldots, i_k \geq 0, r_2 \geq 3$,

$$\mu^{(i_1+1,i_2,\ldots,i_k)}_{1,r_2,\ldots,r_k:n} = (i_1+1)\mu^{(i_1,i_2,\ldots,i_k)}_{1,r_2,\ldots,r_k:n} - nQ_2\Big\{\mu^{(i_1+1,i_2,\ldots,i_k)}_{1,r_2-1,r_3-1,\ldots,r_k-1:n-1} - Q_1^{i_1+1}\mu^{(i_2,i_3,\ldots,i_k)}_{r_2-1,r_3-1,\ldots,r_k-1:n-1}\Big\} + \mu^{(i_1+1,i_2,\ldots,i_k)}_{2,r_2,r_3,\ldots,r_k:n} \qquad (2.3)$$

Proof: Relations in (2.2) and (2.3) may be proved by following exactly the same steps as those in proving Theorem 2.2, which is presented here.

Theorem 2.2. *For* $1 \leq x_1 < x_2 < \ldots < x_k \leq n$, $i_1, i_2, \ldots, i_k \geq 0$, *and* $r_2 = r_1 + 1$,

$$\mu^{(i_1+1,i_2,\ldots,i_k)}_{r_1,r_1+1,r_3,\ldots,r_k:n} = \frac{1}{r_1}\Big[(i_1+1)\mu^{(i_1,i_2,\ldots,i_k)}_{r_1,r_1+1,r_3,\ldots,r_k:n} - nQ_2\Big\{\mu^{(i_1+i_2+1,i_3,\ldots,i_k)}_{r_1,r_3-1,\ldots,r_k-1:n-1} - \mu^{(i_1+1,i_2,\ldots,i_k)}_{r_1-1,r_1,r_3-1,\ldots,r_k-1:n-1}\Big\}\Big] + \mu^{(i_1+i_2+1,i_3,\ldots,i_k)}_{r_1+1,r_3,\ldots,r_k:n} \qquad (2.4)$$

and, for $r_2 - r_1 \geq 2$,

$$\mu^{(i_1+1,i_2,\ldots,i_k)}_{r_1+1,r_2,\ldots,r_k:n} = \frac{1}{r_1}\Big[-(i_1+1)\mu^{(i_1,i_2,\ldots,i_k)}_{r_1,r_2,\ldots,r_k:n} + nQ_2\Big\{\mu^{(i_1+1,i_2,\ldots,i_k)}_{r_1,r_2-1,\ldots,r_k-1:n-1} - \mu^{(i_1+1,i_2,\ldots,i_k)}_{r_1-1,r_2-1,\ldots,r_k-1:n-1}\Big\}\Big] + \mu^{(i_1+1,i_2,\ldots,i_k)}_{r_1,r_2,\ldots,r_k:n} \qquad (2.5)$$

Proof: From equation (2.1), we have for $1 \leq x_1 < x_2 < \ldots < x_k \leq n$

$$\mu^{(i_1,i_2,\ldots,i_k)}_{r_1,r_2,\ldots,r_k:n} = C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{Q_1}^{x_k}\cdots\int_{Q_1}^{x_2} x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k}[F(x_1)]^{r_1-1}[F(x_2)-F(x_1)]^{r_2-r_1-1} \ldots[1-F(x_k)]^{n-r_k}f(x_1)f(x_2)\ldots f(x_k)dx_1dx_2\ldots dx_k$$

$$= C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{Q_1}^{x_k}\cdots\int_{Q_1}^{x_2} x_2^{i_2}x_3^{i_3}\ldots x_k^{i_k}[F(x_3)-F(x_2)]^{r_3-r_2-1}\ldots \ldots[1-F(x_k)]^{n-r_k}I(x_2)f(x_2)\ldots f(x_k)dx_2\ldots dx_k, \qquad (2.6)$$

where

$$I(x_2) = \int_{Q_1}^{x_2} x_1^{i_1}[F(x_1)]^{r_1-1}[F(x_2) - F(x_1)]^{r_2-r_1-1} f(x_1)dx_1. \tag{2.7}$$

Making use of characterizing differential equation (1.3), we have

$$\begin{aligned} I(x_2) &= Q_2 \int_{Q_1}^{x_2} x_1^{i_1}[F(x_1)]^{r_1-1}[F(x_2) - F(x_1)]^{r_2-r_1-1} dx_1 \\ &\quad - \int_{Q_1}^{x_2} x_1^{i_1}[F(x_1)]^{r_1}[F(x_2) - F(x_1)]^{r_2-r_1-1} dx_1 \\ &= Q_2 E(x_2, 1) - E(x_2, 0), \end{aligned} \tag{2.8}$$

where

$$E(x_2, k) = \int_{Q_1}^{x_2} x_1^{i_1}[F(x_1)]^{r_1-k}[F(x_2) - F(x_1)]^{r_2-r_1-1} dx_1; \; k = 0, 1. \tag{2.9}$$

Integrating by parts, treating $x_1^{i_1}$ for integration and rest of the integrand for differentiation, we get for $r_2 - r_1 \geq 2$,

$$\begin{aligned} E(x_2, k) = -\int_{Q_1}^{x_2} \frac{x_1^{i_1+1}}{i_1+1} \{ &(r_1 - k)[F(x_1)]^{r_1-k-1}[F(x_2) - F(x_1)]^{r_2-r_1-1} f(x_1)dx_1 \\ &- (r_2 - r_1 - 1)[F(x_1)]^{r_1-k}[F(x_2) - F(x_1)]^{r_2-r_1-1} f(x_1)dx_1 \} \end{aligned}$$

and, for $r_2 = r_1 + 1$,

$$E(x_2, k) = \frac{1}{i_1+1} \Big[x_2^{i_1+1}[F(x_2)]^{r_1-k} - (r_1 - k) \int_{Q_1}^{x_2} x_1^{i_1}[F(x_1)]^{r_1-k-1} f(x_1)dx_1 \Big].$$

Upon substituting for $E(x_2, 0)$ and $E(x_2, 1)$ in (2.8) and then substituting the resulting expression for $I(x_2)$ in equation (2.6) and simplifying, we derive the relations in (2.4) and (2.5).

Proceeding on similar lines, one can derive the following recurrence relation.

Theorem 2.3. *For* $1 \leq x_1 < x_2 < \ldots < x_k \leq n, \;\; r_k \geq r_{k-1} + 2,$

$$\mu^{(i_1,i_2,\ldots,i_{k-1},i_k+1)}_{r_1,r_2,\ldots,r_{k-1},r_k:n} = \frac{1}{(n-r_k+1)}\Big[(i_k+1)\mu^{(i_1,i_2,\ldots,i_{k-1},i_k)}_{r_1,r_2,\ldots,r_{k-1},r_k:n} - nP_2\Big\{\mu^{(i_1,i_2,\ldots,i_{k-1},i_k+1)}_{r_1,r_2,\ldots,r_{k-1},r_k:n-1} - \mu^{(i_1,i_2,\ldots,i_{k-1},i_k+1)}_{r_1,r_2,\ldots,r_{k-1},r_k-1:n-1}\Big\} + \mu^{(i_1,i_2,\ldots,i_{k-1},i_k+1)}_{r_1,r_2,\ldots,r_{k-1},r_k-1:n}\Big]. \tag{2.10}$$

Proof: From equation (2.1), we have

$$\mu^{(i_1,i_2,\ldots,i_k)}_{r_1,r_2,\ldots,r_k:n} = C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{x_1}^{P_1}\ldots\int_{x_{k-1}}^{P_1} x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k}[F(x_1)]^{r_1-1}[F(x_2)-F(x_1)]^{r_2-r_1-1}\ldots$$
$$\ldots[1-F(x_k)]^{n-r_k}f(x_1)f(x_2)\ldots f(x_k)dx_k dx_{k-1}\ldots dx_2 dx_1$$
$$= C_{r_1,r_2,\ldots,r_k:n}\int_{Q_1}^{P_1}\int_{x_1}^{P_1}\ldots\int_{x_{k-2}}^{P_1} x_1^{i_1}x_2^{i_2}\ldots x_{k-1}^{i_{k-1}}[F(x_1)]^{r_1-1}\ldots[F(x_{k-1})-F(x_{k-2})]^{r_{k-1}-r_{k-2}-1}$$
$$.I(x_{k-1})f(x_1)f(x_2)\ldots f(x_{k-1})dx_{k-1}\ldots dx_1, \tag{2.11}$$

where

$$I(x_{k-1}) = \int_{x_{k-1}}^{P_1} x_k^{i_k}[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-1}[1-F(x_k)]^{n-r_k}f(x_k)dx_k. \tag{2.12}$$

Using the characterizing differential equation (1.4) in (2.12), we get

$$I(x_{k-1}) = P_2\int_{x_{k-1}}^{P_1} x_k^{i_k}[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-1}[1-F(x_k)]^{n-r_k}dx_k$$
$$+\int_{x_{k-1}}^{P_1} x_k^{i_k}[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-1}[1-F(x_k)]^{n-r_k+1}dx_k$$
$$= P_2E(x_{k-1},0) - E(x_{k-1},1) \tag{2.13}$$

where

$$E(x_{k-1},t) = \int_{x_{k-1}}^{P_1} x_k^{i_k}[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-1}[1-F(x_k)]^{n-r_k+t}dx_k\ ;\ t=0,1.$$

Integrating by parts, treating $x_k^{i_k}$ for integration and rest of the integrand for differentiation, we have

$$E(x_{k-1},t) = \frac{(n-r_k+t)}{i_k+1}\int_{x_{k-1}}^{P_1} x_k^{i_k+1}[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-1}[1-F(x_k)]^{n-r_k+t-1}f(x_k)dx_k$$
$$-\frac{(r_k-r_{k-1}-1)}{i_k+1}\int_{x_{k-1}}^{P_1} x_k^{i_k+1}[F(x_k)-F(x_{k-1})]^{r_k-r_{k-1}-2}[1-F(x_k)]^{n-r_k+t}f(x_k)dx_k.$$

Upon substituting for $E(x_{k-1},0)$ and $E(x_{k-1},1)$ in (2.13) and then substituting the resulting expression in (2.11) and simplifying, we derive the relation in (2.10).

Likewise, one can easily derive the recurrence relations given in the following theorem.

Theorem 2.4. *For* $1 \le x_1 < x_2 < \ldots < x_k \le n,\ \ r_k = r_{k-1}+1,$

$$\mu^{(i_1,i_2,\dots,i_{k-1},i_k+1)}_{r_1,r_2,\dots,r_{k-1},r_{k-1}+1:n} = \frac{1}{(n-r_k+1)}\Big[(i_k+1)\mu^{(i_1,i_2,\dots,i_{k-1},i_k)}_{r_1,r_2,\dots,r_{k-1},r_{k-1}+1:n}$$
$$-nP_2\Big\{\mu^{(i_1,i_2,\dots,i_{k-1},i_k+1)}_{r_1,r_2,\dots,r_{k-1},r_{k-1}+1:n-1} - \mu^{(i_1,i_2,\dots,i_{k-1}+i_k+1)}_{r_1,r_2,\dots,r_{k-1}:n-1}\Big\} + \mu^{(i_1,i_2,\dots,i_{k-1}+i_k+1)}_{r_1,r_2,\dots,r_{k-1}:n}\Big] \tag{2.14}$$

and, for $r_k = n$,

$$\mu^{(i_1,i_2,\dots,i_{k-1},i_k+1)}_{r_1,r_2,\dots,r_{k-1},n:n} = (i_k+1)\mu^{(i_1,i_2,\dots,i_{k-1},i_k)}_{r_1,r_2,\dots,r_{k-1},n:n} - nP_2\Big\{P_1^{i_k+1}\mu^{(i_1,i_2,\dots,i_{k-1})}_{r_1,r_2,\dots,r_{k-1}:n-1}$$
$$-\mu^{(i_1,i_2,\dots,i_{k-1},i_k+1)}_{r_1,r_2,\dots,r_{k-1},n-1:n-1}\Big\} + \mu^{(i_1,i_2,\dots,i_{k-1},i_k+1)}_{r_1,r_2,\dots,r_{k-1},n-1:n}. \tag{2.15}$$

Remark 1: By using recurrence relations given in equations (2.4), (2.5), (2.10), (2.14) and (2.15) in a simple recursive way, one can easily obtain all the single, double (product) and higher moments of any order of all order statistics for any sample size from doubly truncated exponential distribution.

Remark 2: Recurrence relations for the single and double (product) moments of order statistics from doubly truncated exponential distribution can be obtained as special cases of the above derived results. Some of the results, so obtained, are in agreement with the results of Joshi (1979) and Balakrishnan and Joshi (1984).

Remark 3: By letting both the proportions of truncation Q and $1 - P \to 0$ ($\Rightarrow Q_2 \to 1, P_2 \to 0$) in Theorems 2.1-2.4, we deduce the corresponding results for standard exponential distribution. Recurrence relations for single and double (product) moments of order statistics from standard exponential distribution can be obtained as particular cases of these results. These results are also in agreement with results of Joshi (1978) for standard exponential distribution.

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