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Asymptotic Behavior of Solution of Cauchy Problem for the Generalized Damped Double Dispersion Equation

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Abstract

This paper studies the Cauchy problem of the generalized damped double dispersion equation. By using multiplier method, it is proven that the global solution of the problem decays to zero exponentially as the time approaches infinite, under a very simple and mild assumption regarding the nonlinear term.

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1 Introduction

In this paper we focus on asymptotic behavior of solutions Cauchy problem of the generalized damped double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = g\left(u\right)_{xx}, \; x \in R, \; t > 0, \tag{1.1}$$

$$u\left(x, 0\right) = \varphi\left(x\right), \; u_t\left(x, 0\right) = \psi\left(x\right), \tag{1.2}$$

where $u(x,t)$ denotes the unknown function, g is a nonlinear function of u, $\varphi(x)$ and $\psi(x)$ are the given initial value functions, α is a constant, the subscripts x and t indicate the partial derivative with respect to x and t. Considering the possibility of energy exchange through lateral surfaces of the waveguide in the physical study of nonlinear wave propagation in waveguide, the longitudinal displacement $u(x,t)$ of the rod satisfies the following double dispersion equation [3, 6, 7]

$$u_{tt} - u_{xx} = \frac{1}{4}\left(6u^2 + au_{tt} - bu_{xx}\right)_{xx},$$

and the general cubic double dispersion equation

$$u_{tt} - u_{xx} = \frac{1}{4}\left(cu^3 + 6u^2 + au_{tt} - bu_{xx} + du_t\right)_{xx},$$

where $a,\ b$ and c are positive constants.

Yacheng and Junsheng [10] studied existence, blow up and asymptotic behavior for class of nonlinear wave equations with dispersive term

$$u_{tt} - u_{xxtt} - \alpha u_{xxt} = \sigma(u_x)_x.$$

The asymptotic behavior and the blow up of solution for a nonlinear evolution equation of fourth order

$$u_{tt} - a_1 u_{xx} - a_2 u_{xxt} - a_3 u_{xxtt} = \varphi(u_x)_x$$

were established in [1].

Polat and Kaya [5] studied the existence, both locally and globally in time, the asymptotic behavior, and the blow up of solution for the class of nonlinear wave equations with dissipative and dispersive terms

$$u_{tt} - u_{xx} - u_{xxtt} - \lambda u_{xxt} + u = \sigma(u_x)_x.$$

Lai and Wu [2] investigated the global solution of the following generalized damped Boussinesq equation

$$u_{tt} - au_{xxtt} - 2bu_{xxt} = -cu_{xxxx} + u_{xx} - p^2 u + \beta\left(u^2\right)_{xx},$$

and obtained the asymptotic expression of the global solution.

Throughout this paper, $H^s = H^s(R)$ will denote the L^2 Sobolev space on R. For the H^s norm we use the Fourier transform representation $\|u\|_{H^s}^2 =$

$\int_R \left(1+\xi^2\right)^s \left|\widehat{u}(\xi)\right|^2 d\xi$. We use $\|u\|_\infty$, $\|u\|$ and (u,v) to denote the L^∞ and L^2 norms and the inner product in L^2, respectively.

Proposition 1.1 [4, 9]. Assume that $s \geq 1,\ \varphi \in H^s,\ \psi \in H^{s-1},\ \wedge^{-1}\psi \in L^2,\ G(\varphi) \in L^1,\ g \in C^{[s]+1}(R)$ and $G(u) \geq 0$ or $g'(u)$ is bounded below, i.e. there is a constant A_0 such that $g'(s) \geq A_0$ for all $s \in R$. Then the problem (1.1), (1.2) has a unique global solution $u \in C\left([0,\infty), H^{s+1}\right) \cap C^1\left([0,\infty), H^s\right) \cap C^2\left([0,\infty), H^{s-1}\right)$.

Where $\wedge^{-\alpha}\psi = F^{-1}\left[|\xi|^{-\alpha} F\psi\right]$, F and F^{-1} denote respectively Fourier transformation and inverse Fourier transformation in R (see [8]), and $G(u) = \int_0^u g(s)\,ds$.

2 Asymptotic Behavior of Solutions

We first prove the following lemma.

Lemma 2.1. If there exist functions $w_0(x) \in H^{s+1}$ and $v_0(x) \in H^s$ such that the initial values $u(x,0)$ and $u_t(x,0)$ satisfy the relations

$$u(x,0) = (w_0(x))_x,\ u_t(x,0) = (v_0(x))_x,$$

then for all $t \in T$ the solution $u(x,t)$ of (1.1), (1.2) satisfies $u(x,t) = (w(x,t))_x$, with a corresponding evolution of $w(x,t),\ v(x,t)$ satisfying the system

$$w_t(x,t) = v(x,t)$$

$$v_t - v_{xxt} - \alpha v_{xx} = w_{xx} - w_{xxxx} + g(w_x)_x. \tag{2.1}$$

Proof. Writing (1.1) in the form

$$u_t = z_x$$

$$z_t - z_{xxt} - \alpha z_{xx} = (u - u_{xx} + g(u))_x, \tag{2.2}$$

and using (2.2) we obtain $u(x,t) = u(x,0) + \int_0^t z_x(x,s)\,ds$. The term $u(x,0)$ is an $x-$derivative by hypothesis and $\int_0^t z_x(x,s)\,ds$ is an $x-$derivative. Therefore there exist a $w(x,t)$ such that $u(x,t) = (w(x,t))_x$ which gives, from (1.1),

$$u = w_x,\ w_{tt} - w_{xx} - w_{xxtt} + w_{xxxx} - \alpha w_{xxt} = g(w_x)_x.$$

Hence we easily obtain the system (2.1).

By the lemma, the problem (1.1), (1.2) corresponding the following problem

$$w_{tt} - w_{xx} - w_{xxtt} + w_{xxxx} - \alpha w_{xxt} = g\left(w_x\right)_x, \ x \in R, \ t > 0, \tag{2.3}$$

$$w\left(x,0\right) = w_0\left(x\right), \ w_t\left(x,0\right) = w_1\left(x\right). \tag{2.4}$$

In this section, we discuss the asymptotic behavior of the solutions for problem (2.3), (2.4). For this purpose we define the energy by

$$E\left(t\right) = \frac{1}{2}\left(\left\|w_t\right\|^2 + \left\|w_x\right\|^2 + \left\|w_{xt}\right\|^2 + \left\|w_{xx}\right\|^2\right) + \int_R G\left(w_x\right) dx, \tag{2.5}$$

where $G\left(s\right) = \int_0^s g\left(y\right) dy$.

Theorem 2.1. Let $\alpha > 0$ and assume that

(i) either $g\left(s\right) s \geq 0$ or $g'\left(s\right) \geq C_0, \ s \in R$, where C_0 is a constant;

(ii) $E\left(0\right) > 0$;

(iii) $G\left(s\right) \leq bg\left(s\right) s, \ s \in R$, where $b > 0$ is a constant.

Then for the global solution $w \in C\left(\left[0,\infty\right), H^{s+1}\right) \cap C^1\left(\left[0,\infty\right), H^s\right) \cap C^2\left(\left[0,\infty\right), H^{s-1}\right)$ of problem (2.3), (2.4) there exist $\delta_1 > 0$ and $C > 0$ such that

$$E\left(t\right) \leq CE\left(0\right) e^{-\delta_1 t}, \ t > 0. \tag{2.6}$$

Proof. Let $w\left(x,t\right)$ be a global solution of problem (2.3), (2.4). Taking the L^2 inner product of (2.3) with w_t it follows that

$$\frac{d}{dt}E\left(t\right) + \alpha\left\|w_{xt}\right\|^2 = 0, \ t > 0. \tag{2.7}$$

Multiplying (2.7) by $e^{\delta t}$ gives

$$\frac{d}{dt}\left(e^{\delta t}E\left(t\right)\right) + \alpha e^{\delta t}\left\|w_{xt}\right\|^2 = \delta e^{\delta t}E\left(t\right), \ t > 0. \tag{2.8}$$

Integrating (2.8) over $\left(0,t\right)$ we get

$$\begin{aligned} e^{\delta t}E\left(t\right) + \alpha\int_0^t e^{\delta\tau}\left\|w_{x\tau}\left(\tau\right)\right\|^2 d\tau &= E\left(0\right) + \delta\int_0^t e^{\delta\tau}E\left(\tau\right) d\tau \\ &= E\left(0\right) + \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\left\|w_\tau\right\|^2 + \left\|w_{x\tau}\right\|^2\right) d\tau \\ &\quad +\delta\int_0^t e^{\delta\tau}\left(\frac{1}{2}\left\|w_x\right\|^2 + \frac{1}{2}\left\|w_{xx}\right\|^2 + \int_R G\left(w_x\right) dx\right) d\tau. \end{aligned} \tag{2.9}$$

Case 1. If $g(s)s \geq 0,\ s \in R$, then $G(s) \geq 0$. Thus from assumption (iii) of Theorem 2.1 we have $0 \leq G(s) \leq bg(s)s$. Using this relation we obtain

$$\begin{aligned}
&\int_0^t e^{\delta\tau}\left(\frac{1}{2}\|w_x\|^2 + \frac{1}{2}\|w_{xx}\|^2 + \int_R G(w_x)\,dx\right)d\tau \\
\leq\ & b_1 \int_0^t e^{\delta\tau}\left(\|w_x\|^2 + \|w_{xx}\|^2 + \int_R g(w_x)\,w_x dx\right)d\tau.
\end{aligned}$$

For the last term, using the integrating by parts and Eq. (2.3) we obtain

$$\begin{aligned}
&= -b_1 \int_0^t e^{\delta\tau}\left(w, w_{\tau\tau} - w_{xx\tau\tau} - \alpha w_{xx\tau}\right)d\tau \\
&= -b_1 \int_0^t e^{\delta\tau}\left((w, w_{\tau\tau}) - (w, w_{xx\tau\tau}) - \alpha(w, w_{xx\tau})\right)d\tau \\
&= -b_1 \int_0^t e^{\delta\tau}\left((w, w_{\tau\tau}) + (w_x, w_{x\tau\tau}) + \frac{\alpha}{2}\|w_x\|^2\right)d\tau.
\end{aligned}$$

Using the integrating by parts, we obtain

$$\begin{aligned}
= & -b_1\left[e^{\delta t}\left((w, w_t) + (w_x, w_{xt}) + \frac{\alpha}{2}\|w_x\|^2\right) - \left((w_0, w_1) + (w_{0x}, w_{1x}) + \frac{\alpha}{2}\|w_{0x}\|^2\right)\right. \\
& \left. - \int_0^t e^{\delta\tau}\left(\|w_\tau\|^2 + \|w_{x\tau}\|^2\right)d\tau + \delta\int_0^t e^{\delta\tau}\left((w, w_\tau) + (w_x, w_{x\tau}) + \frac{\alpha}{2}\|w_x\|^2\right)d\tau\right].
\end{aligned}$$

Using the Young's inequality, we obtain

$$\begin{aligned}
\leq\ & b_1 \int_0^t e^{\delta\tau}\left(\|w_\tau\|^2 + \|w_{x\tau}\|^2\right)d\tau \\
& + \frac{b_1}{2}e^{\delta t}\left(\|w_t\|^2 + \|w\|^2 + \|w_x\|^2 + \alpha\|w_x\|^2 + \|w_{xt}\|^2\right) \\
& + \frac{b_1}{2}\left(\|w_1\|^2 + \|w_0\|^2 + \|w_{0x}\|^2 + \alpha\|w_{0x}\|^2 + \|w_{1x}\|^2\right) \\
& + \frac{b_1}{2}\delta\int_0^t e^{\delta\tau}\left(\|w_\tau\|^2 + \|w\|^2 + \|w_x\|^2 + \alpha\|w_x\|^2 + \|w_{x\tau}\|^2\right)d\tau \\
\leq\ & 2b_1\int_0^t e^{\delta\tau}\|w_{x\tau}\|^2 d\tau + \frac{b_1}{2}e^{\delta t}\left(\|w_t\|^2 + (1+\alpha)\|w_x\|^2 + \|w_{xt}\|^2 + \|w_{xx}\|^2\right) \\
& + \frac{b_1}{2}\left(\|w_1\|^2 + (1+\alpha)\|w_{0x}\|^2 + \|w_{1x}\|^2 + \|w_{0xx}\|^2\right) \\
& + \frac{b_1}{2}\delta\int_0^t e^{\delta\tau}\left(\|w_\tau\|^2 + (1+\alpha)\|w_x\|^2 + \|w_{x\tau}\|^2 + \|w_{xx}\|^2\right)d\tau \\
\leq\ & 2b_1\int_0^t e^{\delta\tau}\|w_{x\tau}\|^2 d\tau + (1+\alpha)\,b_1 e^{\delta t}E(t) \\
& + (1+\alpha)\,b_1 E(0) + (1+\alpha)\,b_1\delta\int_0^t e^{\delta\tau}E(\tau)\,d\tau,\ t > 0, \qquad (2.10)
\end{aligned}$$

where $b_1 = \max\left\{\frac{1}{2}, b\right\}$. Substituting inequality (2.10) into (2.9) we obtain

$$\begin{aligned} e^{\delta t}E(t) + \alpha\int_0^t e^{\delta\tau}\left\|w_{x\tau}(\tau)\right\|^2 d\tau &\leq (1+(1+\alpha)b_1\delta)E(0) \\ &+ (1+2b_1)\delta\int_0^t e^{\delta\tau}\left\|w_{x\tau}(\tau)\right\|^2 d\tau \\ &+ (1+\alpha)b_1\delta e^{\delta t}E(t) \\ &+ (1+\alpha)b_1\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau. \quad (2.11) \end{aligned}$$

Take $\delta:\ 0<\delta<\min\left\{\frac{\alpha}{1+2b_1}, \frac{1}{2(1+\alpha)b_1}\right\}$, we deduce from (2.11) that

$$e^{\delta t}E(t) \leq CE(0) + \theta\delta\int_0^t e^{\delta\tau}E(\tau)\,d\tau, \quad (2.12)$$

where $C = \frac{1+(1+\alpha)b_1\delta}{1-(1+\alpha)b_1\delta}$ and $\theta = \frac{(1+\alpha)b_1\delta}{1-(1+\alpha)b_1\delta}$. Applying the Gronwall inequality to (2.12) we obtain the result of (2.6) for $\delta_1 = (1-\theta)\delta > 0$.

Case 2. If $g'(s) \geq C_0,\ s\in R$, let $\widetilde{g}(s) = g(s) - k_0 s - g(0)$, where $k_0 = \min\{C_0, 0\} \leq 0$. Obviously $\widetilde{g}(0) = 0,\ \widetilde{g}'(s) = g'(s) - k_0 \geq 0,\ \widetilde{g}(s)s \geq 0,\ s\in R$ and if assumption (iii) of Theorem 2.1 holds, then a simple calculation shows that $0 \leq \widetilde{G}(s) = \int_0^s \widetilde{g}(y)\,dy \leq b\widetilde{g}(s)s,\ s\in R$. Therefore substituting $g(s) = \widetilde{g}(s) + k_0 s + g(0)$ into (2.3) and repeating the proof of Case 1 implies the conclusions of Theorem 2.1. The theorem thus is proved.

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