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An Integral Representation of Some k-Hypergeometric Functions

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Abstract

In this paper, we introduce a new and simple integral representation of some k-confluent hypergeometric functions and k-hypergeometric functions.

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1 Introduction

Diaz and Pariguan [1] have introduced and proved some identities of k-gamma function, k-beta function and k-Pochhammer symbol. They have deduced an integral representation of k-gamma function, k-beta function respectively given by

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\tfrac{x}{k}\right) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \quad \operatorname{Re}(x) > 0, \quad k > 0 \tag{1.1}$$

and $$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad x > 0, \quad y > 0. \tag{1.2}$$

They have also provided some useful and applicable relations

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \tag{1.3}$$

$$(x)_{j,k}=\frac{\Gamma_k(x+jk)}{\Gamma_k(x)}; \tag{1.4}$$

where $(x)_{j,k}=x(x+k)(x+2k)...(x+(j-1)k)$, is the k-Pochhammer symbol and

$$\sum_{j=0}^{\infty}(\alpha)_{j,k}\frac{x^j}{j!}=(1-kx)^{-\frac{\alpha}{k}}. \tag{1.5}$$

Recently, Mansour [5] and Kokologiannaki [4] have proved a number of properties and Kokologiannaki has also taken up k-Zeta function

$$\zeta(x,s)=\sum_{j=0}^{\infty}\frac{1}{(x+jk)^s}, k,x>0, s>1. \tag{1.6}$$

The main purpose of this paper is to introduce an integral representation of some k-confluent hypergeometric functions and k-hypergeometric functions so that we can get the usual integral representations discussed in [2, 3], by taking $k\to 1$. We shall use later the following basic results.

$$m^{mj}\left(\frac{x}{m}\right)_{j,k}\left(\frac{x+k}{m}\right)_{j,k}...\left(\frac{x+(m-1)k}{m}\right)_{j,k}=(x)_{mj,k}; \tag{1.7}$$

$$(x)_{mj,k}=\frac{\Gamma_k(x+mjk)}{\Gamma_k(x)}; \tag{1.8}$$

$$\sum_{j=0}^{\infty}\frac{x^j}{j!}=e^x. \tag{1.9}$$

2 Integral Representation of Some k-Confluent Hypergeometric Functions

In this section, we determine integral representations of some k-confluent hypergeometric functions ${}_mF_{m,k}$.

Theorem 2.1:

If $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0,\ k>0,\ m\geq 1,\ m\in Z^+$, then for all finite x

$${}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),...,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),...,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^m}\,dt. \quad (2.1)$$

Proof: First note that for any positive integer j, we get

$$\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}}=\frac{\Gamma_k(\gamma)\Gamma_k(\beta+mjk)}{\Gamma_k(\beta)\Gamma_k(\gamma+mjk)}$$

$$=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta+mjk,\gamma-\beta)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}+mj-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\,dt. \quad (2.2)$$

Now, using Equations (1.7), (1.9) and (2.2), we get

$${}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\sum_{j=0}^{\infty}\frac{\left(\frac{\beta}{m}\right)_{j,k}\left(\frac{\beta+k}{m}\right)_{j,k}\cdots\left(\frac{\beta+k}{m}\right)_{j,k}}{\left(\frac{\gamma}{m}\right)_{j,k}\left(\frac{\gamma+k}{m}\right)_{j,k}\cdots\left(\frac{\gamma+k}{m}\right)_{j,k}}\frac{x^j}{j!}$$

$$=\sum_{j=0}^{\infty}\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}}\frac{x^j}{j!}$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^m}\,dt.$$

Corollary 2.2:
If $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0$, then for all finite x

$${}_1F_{1,k}\left((\beta,k);(\gamma,k);x\right)=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt}\,dt. \quad (2.3)$$

3 Integral Representation of Some k-Hypergeometric Functions

In this section, we determine integral representations of some k-hypergeometric functions.

Theorem 3.1:
If $\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0,\ \ k > 0,\ \ m \geq 1,\ \ m \in Z^{+}$ and $|x| < 1$, then

$$ {}_{m+1}F_{m,k}\left(\begin{matrix} (\alpha,k),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \end{matrix} ;x \right) $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt^m\right)^{-\frac{\alpha}{k}} dt. \qquad (3.1) $$

Proof: First note that for any positive integer j, we have

$$ \frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}} = \frac{\Gamma_k(\gamma)\Gamma_k(\beta+mjk)}{\Gamma_k(\beta)\Gamma_k(\gamma+mjk)} $$

$$ = \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} B_k(\beta+mjk,\gamma-\beta) $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}+mj-1}(1-t)^{\frac{\gamma-\beta}{k}-1} dt. \qquad (3.2) $$

Using Equations (1.7), (3.2) and (1.5), we get

$$ {}_{m}F_{m,k}\left(\begin{matrix} (\alpha,k),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \end{matrix} ;x \right) $$

$$ = \sum_{j=0}^{\infty} \frac{(\alpha)_{j,k}\left(\frac{\beta}{m}\right)_{j,k}\left(\frac{\beta+k}{m}\right)_{j,k}\cdots\left(\frac{\beta+k}{m}\right)_{j,k}}{\left(\frac{\gamma}{m}\right)_{j,k}\left(\frac{\gamma+k}{m}\right)_{j,k}\cdots\left(\frac{\gamma+k}{m}\right)_{j,k}} \frac{x^j}{j!} $$

$$ = \sum_{j=0}^{\infty} \frac{(\alpha)_{j,k}(\beta)_{mj,k}}{(\gamma)_{mj,k}} \frac{x^j}{j!} $$

$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt^m\right)^{-\frac{\alpha}{k}} dt .$$

Corollary 3.1:

If $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0$ and $|x| < 1$, then

$${}_2F_{1,k}\left((\alpha,k),(\beta,k);(\gamma,k);x\right)$$

$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt\right)^{-\frac{\alpha}{k}} dt . \qquad (3.3)$$

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