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Convergence of Lagrange Interpolation on the Unit Circle

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Abstract: In this paper, we study the convergence of Lagrange interpolation polynomials on the sets obtained by projecting vertically the zeros of $(1-x^2)\,P_n(x)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial.

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1. Introduction:

In a paper, P. Nevai [6]considered the necessary and sufficient conditions for convergence of Lagrange interpolation based at the zeros of generalized Jacobi polynomials in Lp spaces with general weights. In 1994, Xin Li and E.B. Saff [3] considered the function f which is bounded on $[-1,1]$ and analytic at $x=0$ and proved the local convergence of Lagrange interpolating polynomials of f associated with equidistant nodes on $[-1,1]$. In 1997, F. Peherstarfer [7] have studied the convergence of Lagrange interpolation on the triangular matrix X given by $-1 < x_{1.n} < x_{2,n} < ...... < x_{n,n} < 1, n = 1, 2, ...$

After that, D.S. Lubinsky [4] obtained the necessary and sufficient condition for the mean convergence of the Lagrange interpolation at the zeros of orthogonal polynomials for weights on $[-1,1]$. In another paper [5] , he considered the mean convergence of Lagrange interpolating polynomial. Later on S.B. Damelin, H.S. Jung and K.H. Kwan [2] considered the mean convergence of Lagrange interpolation for weights in $L_p\,(0 < p < 1)$.

In the case of complex plane, R. Brűck [1] have studied the convergence of Lagrange interpolation of a function on the nodes $z_{kn}^{\alpha} = T_{\alpha}\,(\omega_{kn})\,, k =$

$1\,(1)\,2n$ where T_α is a Mőbius transform of a unit disk into itself and $\omega_{kn} = \exp\left(\frac{2\pi ik}{2n+1}\right), n \geqslant 0$. Recently Xie Siqing [9] cosidered weighted least square convergence of Lagrange interpolation polynomials based on the sets obtained by projecting vertically the zeros of
$(1-x^2)\,P_n^{(\alpha,\beta)}(x)\,(\alpha>0,\beta>0)\,,(1+x)\,P_n^{(\alpha,\beta)}(x)\,(\alpha>-1,\beta>0)\,,(1-x)\,P_n^{(\alpha,\beta)}(x)$ $(\alpha>0,\beta>-1)$ and $P_n^{(\alpha,\beta)}(x)$ $(\alpha>-1,\beta>-1)$, respectively onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ $(\alpha>-1,\beta>-1)$ stands for n^{th} Jacobi polynomial.

The Problem:

In this paper, we have considered the congervence of Lagrange interpolatoion polynomials on the nodes obtained by projecting vertically the zeros of $(1-x^2)\ P_n(x)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial.

Let

(1.1) $Z_n = \{z_0 = 1,\ z_{2n+1} = -1,\ z_k = \cos\theta_k + i\sin\theta_k,\ z_{n+k} = -z_k\ ,\ \ k = 1\,(1)\,n\}$

be the zeros of

(1.2) $$R(z) = (z^2-1)\,W(z)$$

where

(1.3) $$W(z) = \prod_{k=1}^{2n}(z-z_k) = K_n P_n(x)\,z^n$$

Then Z_n denotes the vertical projections on the unit circle of the zeros of $(1-x^2)\ P_n(x)$. The object of this paper is to obtain a quantitative estimate of $|L_n(z) - f(z)|$, where $L_n(z)$ is the Lagrange interpolation polynomial of degree $\leq 2n+1$. Then

(1.4) $$L_n(z) = \sum_{k=0}^{2n+1} f\ (z_k)\,l_k(z)$$

where

(1.5) $$l_k(z) = \frac{R(z)}{R'(z_k)(z-z_k)}$$

such that

(1.6) $$l_k(z_j) = \delta_{kj} = \begin{cases} 0, & k \neq j \\ 1, & k = j \end{cases},\ \text{for } j,k = 0\,(1)\,2n+1\ .$$

2. Main Result:

Let λ_n be the Lebesgue constant for l_k *i.e.*

(2.1) $$\lambda_n = \max\ \lambda_n(z) \text{ for } |z| \leq 1$$

where

$$(2.2) \qquad \lambda_n(z) = \sum_{k=0}^{2n+1} |l_k(z)|\,.$$

Lemma 1:$[8, 694 - 695]$

$$(2.3) \qquad \lambda_n \leq c_1 \log n + c_{2,}$$

where c_1 and c_2 are constants.

Theorem: Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$, then the sequence of interpolatory polynomial $\{\ L_n\}$ defined by (1.4) satisfies the relation

$$(2.4) \qquad |L_n(z) - f(z)| = o(1)\,.$$

Remark: To prove the theorem .we need Jackson's theorem:

Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$, then there exists a polynomial $F_n(z)$ of degree $\leq 2n+1$ such that

$$(2.5) \qquad |F_n(z) - f(z)| \leq c\omega\left(f.\tfrac{1}{n}\right),\ z = e^{i\theta}, (\ 0 \leq \theta < 2\pi),$$

where $\omega(f.\delta)$ is the modulus of smoothness of $f(z)$.

Proof: Since $L_n(z)$ is the uniquely determined polynomial of degree $\leq 2n+1$ given by (1.4) ,therefore, the polynomial $F_n(z)$ satisfying (2.5) can be expressed as:

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k)\, l_k(z)\,.$$

Then

$$|L_n(z) - f(z)| \leq |L_n(z) - F_n(z)| + |F_n(z) - f(z)|$$

Taking $z = e^{i\theta}, (\ 0 \leq \theta < 2\pi)$ and using (2.3) and (2.5), we get (2.4), which completes the proof of the theorem.

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