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On Ramanujan's ${}_1\psi_1$ Summation Formula

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Abstract

In this paper, we present the Ramanujan's ${}_1\psi_1$ summation formula and Heine's Identity. Some identities which corresponding to Ramanujan's ${}_1\psi_1$ summation formula and Heine's Identity are presented.

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1 Introduction

In H. Prodinger's note [1] use identity:

$$\frac{(a;q)_\infty}{(a;q^2)_\infty} = (aq;q^2)_\infty.$$

In this study we proof this identity and the generalization of this identity. The Ramanujan's ${}_1\psi_1$ summation formula:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(b/a;q)_\infty (ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/ax;q)_\infty (b;q)_\infty (x;q)_\infty}; \qquad |q|<1, |\frac{b}{a}|<|x|<1$$

is presented in H. J. Lam [3] and the proof of Ramanujan's ${}_1\psi_1$ summation formula is presented in his thesis. In [1]and [2], Heine's identity is presented. In this study, we would like to obtain some identities for Ramanujan's ${}_1\psi_1$ summation formula and Heine's identity without the proof of converge of the series.

2 Results

2.1 Notation of Products

Let τ be a fixed complex number satisfying $Im\tau > 0$ and let $q = e^{i\pi\tau}$, so that $|q| < 1$. We will make use of the following notation for products. Let

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \qquad n = 1, 2, 3, \dots \tag{1}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{2}$$

Then

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{3}$$

and we take this as a definition for $n \in$ R. We define

$$(a_1, a_2, \dots, a_n; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \dots (a_n;q)_\infty. \tag{4}$$

Since

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

and

$$(aq^n;q)_\infty = \prod_{k=0}^{\infty}(1-aq^nq^k) = \prod_{k=0}^{\infty}(1-aq^{n+k}).$$

Then

$$\begin{aligned}
\frac{(a;q)_\infty}{(aq^n;q)_\infty} &= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^{n+k})} \\
&= \frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)\dots}{(1-aq^n)(1-aq^{n+1})(1-aq^{n+2})(1-aq^{n+3})\dots} \\
&= (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}) \\
&= \prod_{k=0}^{n-1}(1-aq^k) \\
&= (a;q)_n.
\end{aligned}$$

Therefore equality (3) is proved.
From (2), we have

Proposition 2.1.1

$$\frac{(a;q)_\infty}{(a;q^2)_\infty} = (aq;q^2)_\infty.$$

Proof.

$$\begin{aligned}
\frac{(a;q)_\infty}{(a;q^2)_\infty} &= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-a(q^2)^k)} \\
&= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^{2k})} \\
&= \frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots}{(1-a)(1-aq^2)(1-aq^4)(1-aq^6)\ldots} \\
&= (1-aq)(1-aq^3)(1-aq^5)\ldots \\
&= \prod_{k=0}^{\infty}(1-aq^{2k+1}) \\
&= \prod_{k=0}^{\infty}(1-aq(q^2)^k) \\
&= (aq;q^2)_\infty.
\end{aligned}$$

Proposition 2.1.2

$$\frac{(a;q)_\infty}{(a;q^3)_\infty} = (aq;q^3)_\infty(aq^2;q^3)_\infty.$$

Proof.

$$\begin{aligned}
\frac{(a;q)_\infty}{(a;q^3)_\infty} &= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-a(q^3)^k)} \\
&= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^{3k})} \\
&= \frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots}{(1-a)(1-aq^3)(1-aq^6)(1-aq^9)\ldots} \\
&= (1-aq)(1-aq^2)(1-aq^4)(1-aq^5)(1-aq^7)(1-aq^8)\ldots \\
&= \prod_{k=0}^{\infty}(1-aq^{3k+1})\prod_{k=0}^{\infty}(1-aq^{3k+2}) \\
&= \prod_{k=0}^{\infty}(1-aqq^{3k})\prod_{k=0}^{\infty}(1-aq^2q^{3k}) \\
&= (aq;q^3)_\infty(aq^2;q^3)_\infty.
\end{aligned}$$

We have the generalize of Proposition 2.1 and Proposition 2.2 as following:

Theorem 2.1.3

$$\frac{(a;q)_\infty}{(a;q^n)_\infty} = (aq;q^n)_\infty(aq^2;q^n)_\infty \ldots (aq^{n-1};q^n)_\infty \qquad n \geq 2.$$

Proof.

$$\begin{aligned}
\frac{(a;q)_\infty}{(a;q^n)_\infty} &= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-a(q^n)^k)} \\
&= \frac{\prod_{k=0}^{\infty}(1-aq^k)}{\prod_{k=0}^{\infty}(1-aq^{nk})} \\
&= \frac{(1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots}{(1-a)(1-aq^n)(1-aq^{2n})(1-aq^{3n})\ldots} \\
&= (1-aq)(1-aq^2)(1-aq^4)\ldots(1-aq^{n-1})(1-aq^{n+1})\ldots \\
&\quad (1-aq^{2n-1})(1-aq^{2n+1})\ldots \\
&= \prod_{k=0}^{\infty}(1-aq^{nk+1})\prod_{k=0}^{\infty}(1-aq^{nk+2})\cdots\prod_{k=0}^{\infty}(1-aq^{nk+(n-1)}) \\
&= \prod_{k=0}^{\infty}(1-aqq^{nk})\prod_{k=0}^{\infty}(1-aq^2q^{nk})\cdots\prod_{k=0}^{\infty}(1-aq^{n-1}q^{nk}) \\
&= (aq;q^n)_\infty(aq^2;q^n)_\infty\ldots(aq^{n-1};q^n)_\infty.
\end{aligned}$$

2.2 Ramanujan's ${}_1\psi_1$ summation formula and some identities

Ramanujan's identity is:

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n = \frac{(b/a;q)_\infty(ax;q)_\infty(q/ax;q)_\infty(q;q)_\infty}{(q/a;q)_\infty(b/ax;q)_\infty(b;q)_\infty(x;q)_\infty}; \qquad |q|<1, |\frac{b}{a}|<|x|<1 \tag{5}$$

We not show the prove of Ramanujan's identity. The proof of this identity is shown in H. J. Lam [3]. There are different identities which corresponding to Ramanujan's summation such as the following:

Proposition 2.2.1:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}; \qquad |q|<1, |x|<1 \tag{6}$$

Proof. The prove of this proposition we set $b = q$ in (5). This proposition is called *q-series* or *Non-terminating q-binomial Theorem.*

Proposition 2.2.2:

$$\sum_{n=0}^{\infty} \frac{(q^k;q)_n}{(q;q)_n} x^n = \frac{1}{(x;q)_\infty}; \qquad |q| < 1, |x| < 1 \tag{7}$$

Proof. The prove of this corollary we set $b = q$ and $a = q^k$ in (5) or set $a = q^k$ in (6).

Proposition 2.2.3:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}; \qquad |x| < 1. \tag{8}$$

Proof. The prove of this corollary we set $b = q$ and $a = q$ in (5) or set $a = q^k$ in (6) and set $k = 1$ in (7).
We see that proposition 2.2.3 is the form geometric series.

Proposition 2.2.4:

$$-ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}; \qquad |x| < 1. \tag{9}$$

Proof. By proposition 2.2.3, we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}; \qquad |x| < 1$$

Integrating two side of the previous equation with respect to x, we have that

$$\int_{n=0}^{\infty} \sum_{n=0}^{\infty} x^n dx = \int_{n=0}^{\infty} \frac{1}{1-x} dx$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -ln(1-x).$$

Therefore proposition 2.2.4 is proved.

Proposition 2.2.5:

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}; \qquad |q| < 1, |x| < 1 \tag{10}$$

Proof. By proposition 2.2.1, we have

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}; \qquad |q| < 1, |x| < 1.$$

Take $a = 0$, then

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}.$$

Therefore proposition 2.2.5 is proved.

Theorem 2.2.6:

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^n} = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty^2}{(q/a;q)_\infty (q/x;q)_\infty (a;q)_\infty (x;q)_\infty}; \qquad |q| < |x| < 1. \tag{11}$$

Proof. From Ramanujan's summation formula, we have:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(b/a;q)_\infty (ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b/ax;q)_\infty (b;q)_\infty (x;q)_\infty}; \qquad |q| < 1, |\frac{b}{a}| < |x| < 1.$$

Take $b = aq$:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(aq;q)_n} x^n = \frac{(q;q)_\infty (ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty}{(q/a;q)_\infty (q/x;q)_\infty (aq;q)_\infty (x;q)_\infty}; \qquad |q| < |x| < 1.$$

Divide by $1 - a$:

$$\sum_{n=-\infty}^{\infty} \frac{1}{1-aq^n} x^n = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty^2}{(q/a;q)_\infty (q/x;q)_\infty (a;q)_\infty (x;q)_\infty}; \qquad |q| < |x| < 1.$$

Then this proposition is proved.

2.3 Heine's Identities and some identities

Heine's identity is:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} x^n = \frac{(b;q)_\infty (ax;q)_\infty}{(c;q)_\infty (x;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_n (x;q)_n}{(q;q)_n (ax;q)_n} b^n \tag{12}$$

Proof. See in [1, 2] .

Lemma 2.3.1:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} x^n = \frac{(c/b;q)_\infty(bx;q)_\infty}{(c;q)_\infty(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(abx/c;q)_n(b;q)_n}{(q;q)_n(bx;q)_n} b^n \tag{13}$$

Proof. See in [2] .

Lemma 2.3.2:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n + \sum_{n=1}^{\infty} \frac{(q/b;q)_n}{(q/a;q)_n} (b/ax)^n \tag{14}$$

Proof. See in [1, 2] .

Theorem 2.3.3: If $|q| < \min\{1, |b|\}$ and $|\frac{b}{a}| < |x| < 1$, then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n &= -1 + \frac{(b/a;q)_\infty(ax;q)_\infty}{(b;q)_\infty(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n(aqx/b;q)_n}{(q;q)_n(ax;q)_n} (b/a)^n \\ &\quad + \frac{(q/b;q)_\infty(bq/ax;q)_\infty}{(q/a;q)_\infty(b/ax;q)_\infty} \sum_{n=0}^{\infty} \frac{(b/a;q)_n(b/ax;q)_n}{(q;q)_n(bq/ax;q)_n} (q/b)^n. \end{aligned} \tag{15}$$

Proof. From Lemma 2.3.2, we have:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n + \sum_{n=1}^{\infty} \frac{(q/b;q)_n}{(q/a;q)_n} (b/ax)^n \\ &= -1 + \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n + \sum_{n=0}^{\infty} \frac{(q/b;q)_n}{(q/a;q)_n} (b/ax)^n. \end{aligned} \tag{16}$$

Take $a = q, b = a$ and $c = b$ in (13), we have that:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(b/a;q)_\infty(ax;q)_\infty}{(b;q)_\infty(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n(aqx;q)_n}{(q;q)_n(ax;q)_n} (b/a)^n. \tag{17}$$

Changing x to b/ax, a to q, b to q/b and c to q/a in (12), we have that:

$$\sum_{n=0}^{\infty} \frac{(q/b;q)_n}{(q/a;q)_n} (b/ax)^n = \frac{(q/b;q)_\infty(bq/ax;q)_\infty}{(q/a;q)_\infty(b/ax;q)_\infty} \sum_{n=0}^{\infty} \frac{(b/a;q)_n(b/ax;q)_n}{(q;q)_n(bq/ax;q)_n} (q/b)^n. \tag{18}$$

Substituting (17) and (18) into (16), we have that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n &= -1 + \frac{(b/a;q)_\infty(ax;q)_\infty}{(b;q)_\infty(x;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n(aqx/b;q)_n}{(q;q)_n(ax;q)_n} (b/a)^n \\ &\quad + \frac{(q/b;q)_\infty(bq/ax;q)_\infty}{(q/a;q)_\infty(b/ax;q)_\infty} \sum_{n=0}^{\infty} \frac{(b/a;q)_n(b/ax;q)_n}{(q;q)_n(bq/ax;q)_n} (q/b)^n. \end{aligned}$$

Therefore the proof is completed.

3 Conclusion

In this research we have some identities which corresponding to Ramanujan's ${}_1\psi_1$ summation formula.We have that the approximation of $-ln(1-x)$ in proposition 2.2.4. The finally we have an identity which corresponding to Heine's identity and Ramanujan's ${}_1\psi_1$ summation formula in Theorem 2.3.3.

References

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