International Mathematical Forum, 4, 2009, no. 7, 303 - 308

Identities for the Common Factors of Fibonacci and Lucas numbers

Montri Thongmoon

Department of Mathematics, Faculty of Science
Mahasarakham University, Mahasarakham, 44150, Thailand
montri.t@msu.ac.th

Abstract

In this paper, we obtain the identities for the common factors of Fibonacci and Lucas numbers. New identities for even and odd Fibonacci numbers are obtained.

Mathematics Subject Classification: 11B37, 11B39

Keywords: Fibonacci numbers, Lucas numbers, Recurrence relation

1 Introduction

It is well-known that the Fibonacci and Lucas numbers are given the recurrence relation $F_{n+1} = F_n + F_{n-1}$, where $n \geq 1$ with the initial conditions $F_0 = 0, F_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$, where $n \geq 1$ with the initial conditions $L_0 = 2, L_1 = 1$.

The fibonacci numbers can be written in the general form as:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \text{ where } n \geq 1 \tag{1}$$

The relation in (1) is called Binet's formula. The Lucas numbers can be written in the general form as:

$$L_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n \text{ where } n \geq 1. \tag{2}$$

There are a lot of identities about the Fibonacci and Lucas numbers. We obtain a new identities for the common factors of Fibonacci and Lucas numbers.

2 Identities for the common factors of Fibonacci and Lucas numbers

Theorem 1 $F_{4n} + 1 = F_{2n-1}L_{2n+1}$ *where* $n \geq 1$.

Proof.

$$
\begin{aligned}
F_{2n-1}L_{2n+1} &= [\tfrac{(1+\sqrt{5})^{2n-1}-(1-\sqrt{5})^{2n-1}}{2^{2n-1}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{2n+1} + (\tfrac{1-\sqrt{5}}{2})^{2n+1}] \\
&= [\tfrac{(1+\sqrt{5})^{2n-1}-(1-\sqrt{5})^{2n-1}}{2^{2n-1}\sqrt{5}}][(\tfrac{(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}}{2^{2n+1}}] \\
&= F_{4n} + [\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\
&= F_{4n} + (-1)^{2n} \\
&= F_{4n} + 1
\end{aligned}
$$

■

Theorem 2 $F_{4n+1} + 1 = F_{2n+1}L_{2n}$ *where* $n \geq 1$.

Proof.

$$
\begin{aligned}
F_{2n-1}L_{2n} &= [\tfrac{(1+\sqrt{5})^{2n-1}-(1-\sqrt{5})^{2n-1}}{2^{2n-1}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{2n} + (\tfrac{1-\sqrt{5}}{2})^{2n}] \\
&= [\tfrac{(1+\sqrt{5})^{2n-1}-(1-\sqrt{5})^{2n-1}}{2^{2n-1}\sqrt{5}}][(\tfrac{(1+\sqrt{5})^{2n}+(1-\sqrt{5})^{2n}}{2^{2n}}] \\
&= F_{4n+1} + [\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\
&= F_{4n+1} + (-1)^{2n} \\
&= F_{4n+1} + 1
\end{aligned}
$$

■

Theorem 3 $F_{4n+2} + 1 = F_{2n+2}L_{2n}$ *where* $n \geq 1$.

Proof.

$$\begin{aligned}
F_{2n+2}L_{2n} &= [\tfrac{(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}}{2^{2n+2}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{2n}+(\tfrac{1-\sqrt{5}}{2})^{2n}] \\
&= [\tfrac{(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}}{2^{2n+2}\sqrt{5}}][(\tfrac{(1+\sqrt{5})^{2n}+(1-\sqrt{5})^{2n}}{2^{2n}}] \\
&= F_{4n+2}+[\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\
&= F_{4n+2}+(-1)^{2n} \\
&= F_{4n+2}+1
\end{aligned}$$

■

By the same way, we have the following result:

Theorem 4 $F_{4n+3}+1=F_{2n+1}L_{2n+2}$ *where* $n \geq 1$*.*

Lemma 5 $F_nL_n=F_{2n}$ *where* $n \geq 1$*.*

Proof.

$$\begin{aligned}
F_nL_n &= [\tfrac{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}}{2^{n}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{n}+(\tfrac{1-\sqrt{5}}{2})^{n}] \\
&= [\tfrac{(1+\sqrt{5})^{n}}{2^{n}\sqrt{5}}-\tfrac{(1-\sqrt{5})^{n}}{2^{n}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{n}+(\tfrac{1-\sqrt{5}}{2})^{n}] \\
&= [\tfrac{(1+\sqrt{5})^{2n}}{2^{2n}\sqrt{5}}-\tfrac{(1-\sqrt{5})^{2n}}{2^{2n}\sqrt{5}}] \\
&= [\tfrac{(1+\sqrt{5})^{2n}-(1-\sqrt{5})^{2n}}{2^{2n}\sqrt{5}}] \\
&= F_{2n}
\end{aligned}$$

■

Lemma 6 $F_{4n+1}-1=F_{2n}L_{2n+1}$ *where* $n \geq 1$*.*

Proof.

$$\begin{aligned}
F_{2n}L_{2n+1} &= [\tfrac{(1+\sqrt{5})^{2n}-(1-\sqrt{5})^{2n}}{2^{2n}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{2n+1}+(\tfrac{1-\sqrt{5}}{2})^{2n+1}] \\
&= [\tfrac{(1+\sqrt{5})^{2n}-(1-\sqrt{5})^{2n}}{2^{2n}\sqrt{5}}][(\tfrac{(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}}{2^{2n+1}}] \\
&= F_{4n+1}-[\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\
&= F_{4n+1}-(-1)^{2n} \\
&= F_{4n+1}-1
\end{aligned}$$

■

From Lemma 5 and Lemma 6, we have the following result:

Corollary 7 $F_{4n+1} - 1 = F_n L_n L_{2n+1}$ *where* $n \geq 1$.

Lemma 8 $L_{4n+1} - 1 = 5F_{2n}F_{2n+1}$ *where* $n \geq 1$.

Proof.

$$\begin{aligned} 5F_{2n}F_{2n+1} &= 5[\tfrac{(1+\sqrt{5})^{2n}-(1-\sqrt{5})^{2n}}{2^{2n}\sqrt{5}}][\tfrac{(1+\sqrt{5})^{2n+1}-(1-\sqrt{5})^{2n+1}}{2^{2n+1}\sqrt{5}}] \\ &= [\tfrac{(1+\sqrt{5})^{4n+1}+(1-\sqrt{5})^{4n+1}}{2^{4n+1}}] - [(\tfrac{(1+\sqrt{5})^{2n+1}(1-\sqrt{5})^{2n}+(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n+1}}{2^{4n+1}}] \\ &= L_{4n+1} - [\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\ &= L_{4n+1} - (-1)^{2n} \\ &= L_{4n+1} - 1 \end{aligned}$$

■

Theorem 9 $L_{4n+1} - 1 = 5F_n L_n F_{2n+1}$ *where* $n \geq 1$.

Proof. Substituting $F_{2n} = F_n L_n$ from Lemma 5 into Lemma 8, then the prove is completed. ■

Theorem 10 $L_{4n+1} + 1 = L_{2n}L_{2n+1}$ *where* $n \geq 1$.

Proof.

$$\begin{aligned} L_{2n}L_{2n+1} &= [(\tfrac{1+\sqrt{5}}{2})^{2n} + (\tfrac{1-\sqrt{5}}{2})^{2n}][(\tfrac{1+\sqrt{5}}{2})^{2n+1} + (\tfrac{1-\sqrt{5}}{2})^{2n+1}] \\ &= [\tfrac{(1+\sqrt{5})^{4n+1}+(1-\sqrt{5})^{4n+1}}{2^{4n+1}}] + [(\tfrac{(1+\sqrt{5})^{2n+1}(1-\sqrt{5})^{2n}+(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n+1}}{2^{4n+1}}] \\ &= L_{4n+1} + [\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\ &= L_{4n+1} + (-1)^{2n} \\ &= L_{4n+1} + 1 \end{aligned}$$

■

Lemma 11 $F_{4n+3} - 1 = F_{2n+2}L_{2n+1}$ *where* $n \geq 1$.

Proof.

$$\begin{aligned}
F_{2n+2}L_{2n+1} &= [\tfrac{(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}}{2^{2n+2}\sqrt{5}}][(\tfrac{1+\sqrt{5}}{2})^{2n+1}+(\tfrac{1-\sqrt{5}}{2})^{2n+1}] \\
&= [\tfrac{(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}}{2^{2n+2}\sqrt{5}}][(\tfrac{(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}}{2^{2n+1}}] \\
&= F_{4n+3}-[\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\
&= F_{4n+3}-(-1)^{2n} \\
&= F_{4n+3}-1
\end{aligned}$$

■

Theorem 12 $F_{4n+3}-1=F_{n+1}L_{n+1}L_{2n+1}$ *where* $n\geq 1$.

Proof. Substituting $F_{2n+2}=F_{n+1}L_{n+1}$ from Lemma 5 into Lemma 11, then the prove is completed. ■

Theorem 13 $L_{4n+3}-1=L_{2n+1}L_{2n+2}$ *where* $n\geq 1$.

Proof.

$$\begin{aligned}
L_{2n+1}L_{2n+2} &= [(\tfrac{1+\sqrt{5}}{2})^{2n+1}+(\tfrac{1-\sqrt{5}}{2})^{2n+1}][(\tfrac{1+\sqrt{5}}{2})^{2n+2}+(\tfrac{1-\sqrt{5}}{2})^{2n+2}] \\
&= [\tfrac{(1+\sqrt{5})^{4n+3}+(1-\sqrt{5})^{4n+3}}{2^{4n+3}}]+[(\tfrac{(1+\sqrt{5})^{2n+2}(1-\sqrt{5})^{2n+1}+(1+\sqrt{5})^{2n+1}(1-\sqrt{5})^{2n+2}}{2^{4n+3}}] \\
&= L_{4n+3}+[\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n+1} \\
&= L_{4n+3}+(-1)^{2n+1} \\
&= L_{4n+3}-1
\end{aligned}$$

■

Since $F_{2n+2}=F_{n+1}L_{n+1}$ where $n\geq 1$. Then we have the following result:

Lemma 14 $L_{4n+3}+1=5F_{2n+2}F_{2n+1}$ *where* $n\geq 1$.

Proof.

$$\begin{aligned}
5F_{2n+2}F_{2n+1} &= 5[\tfrac{(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}}{2^{2n+2}\sqrt{5}}][\tfrac{(1+\sqrt{5})^{2n+1}-(1-\sqrt{5})^{2n+1}}{2^{2n+1}\sqrt{5}}] \\
&= [\tfrac{(1+\sqrt{5})^{4n+3}-(1-\sqrt{5})^{4n+3}}{2^{4n+3}}][(\tfrac{(1-\sqrt{5})^{2n+2}(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}(1+\sqrt{5})^{2n+2}}{2^{4n+3}}] \\
&= L_{4n+3}+[\tfrac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}]^{2n} \\
&= L_{4n+3}+(-1)^{2n} \\
&= L_{4n+3}+1
\end{aligned}$$

■

Substituting $F_{2n+2} = F_{n+1}L_{n+1}$ into Lemma 14, then we have the following result:

Corollary 15 $L_{4n+3} + 1 = 5L_{n+1}F_{n+1}F_{2n+1}$ *where* $n \geq 1$.

References

[1] H. Dubner and W. Keller, New Fibonacci and Lucas primes, Mathematics of Computation, **68**(1999), 417-427.

[2] T. Koshy, Fibonacci and Lucas Numbers with Applications, A Wiley- Interscience Publication, New York, 2001.

Received: September 23, 2008