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Composing Circular Arcs with Minimal NURBS

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Abstract

We study the problem of constructing a NURBS (nonuniform rational B-spline curve) with a minimal number of knots forming a given set of continuously resp. smoothly connected circular arcs. In the continuous case, NURBS defined by a minimal number of conditions can always be constructed. In the case when the circular arcs are smoothly connected, we give a characterization of the existence of minimal NURBS.

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1 Introduction

The volutes at Ionic pillar capitals consist of circular arcs which form a spiral curve within a given frame. This interesting detail of the antique architecture has been investigated by architects, archaeologists and other experts in many papers (s. [1], [3] and the literature therein).

In Fig. 1 we have such a volute with 3 rotations generated by 12 circular arcs, and the points of contact. The centres of the arcs are situated within an

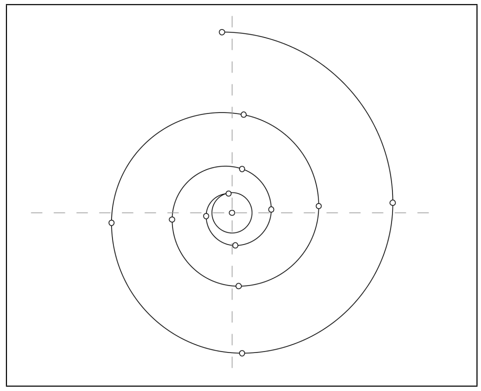

Figure 1: Volute with 3 rotations and eye

eye on a grid parallel to the axes. They are the vertices of the polygon in Fig. 2.

Usually, in the antique architecture a tangential transition has been chosen between circular arcs, i.e., both centres of the corresponding arcs and the point of contact are collinear. But to obtain a regular geometric structure of the points of contact certain transitions of the arcs are only "almost tangential", and the centres of the corresponding arcs and the point of contact do not lie on a straight line (s. [1], [3]).

Since conic sections can be represented by rational Bézier-curves of degree two, it suggests itself to describe connected curves, generated by composed circular arcs using nonuniform rational B-splines, the so-called NURBS, of degree two. In this paper we show how to choose a minimal number of control points and knots in order to construct a NURBS forming continuously resp. smoothly connected circular arcs.

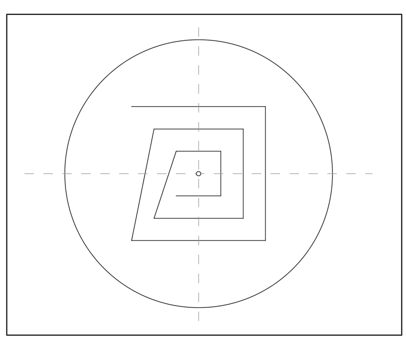

Figure 2: Centres of the arcs from Fig. 1

Since we may assume that the arcs have pairwise different radii or centres, for each arc we need at least one knot interval to represent the arc as a NURBS segment. It turns out that the construction of the NURBS is always possible, if we allow double knots. Hence the question arises to give a lower limit for the necessary number of knots to construct the corresponding NURBS. We first answer this question for the case of circular arcs where the changes in adjacent arcs are assumed to be continuous only.

The case of arcs with smooth transitions takes some more effort. We first give a lower bound for the number of conditions to construct the corresponding NURBS. In order to show that this number is sufficient, a nonlinear system of equations has to be solved to determine the weights and knots. Hence we have to show the existence of a rational spline curve with variable knots. For the case of two composed circular arcs, we always obtain a solution. Unfortunately, this is no longer true for $L > 2$ (L denotes the number of arcs). However, we are able to establish a characterization of the existence of solutions for every $L > 2$. For some small numbers L, we give the exact formulas for the weights and knots.

2 The Problem

Assume that L ($L \geq 2$) circular arcs $K_1, \ldots, K_L$ are given in $\mathbb{R}^2$. Moreover, assume that each K_i belongs to a circle with centre $\mathbf{p}_i$ and radius r_i. Let $\mathbf{d}_{i-1}$ and $\mathbf{d}_i$ be the endpoints of K_i, and suppose that

$$0 < \alpha_i < \pi \tag{2.1}$$

for the angle $\alpha_i = \sphericalangle(\mathbf{d}_{i-1}, \mathbf{p}_i, \mathbf{d}_i)$ at $\mathbf{p}_i$, $i = 1, \ldots, L$.

We consider the tangents at K_i in $\mathbf{d}_{i-1}$ and $\mathbf{d}_i$, and denote their intersection point by $\mathbf{c}_i$, $i = 1, \ldots, L$. These points exist by (2.1), and satisfy the conditions

$$\|\mathbf{d}_{i-1} - \mathbf{c}_i\| = \|\mathbf{d}_i - \mathbf{c}_i\|, \quad i = 1, \ldots, L, \tag{2.2}$$

where $\| \cdot \|$ denotes the Euclidean norm (s. Fig. 3).

Moreover, we assume that

$$K_i \cup K_{i+1} \text{ does } \textbf{not} \text{ form a circular arc for } i = 1, \ldots, L-1. \tag{2.3}$$

Setting

$$C = \bigcup_{i=1}^{L} K_i$$

we obtain a connected curve in $\mathbb{R}^2$.

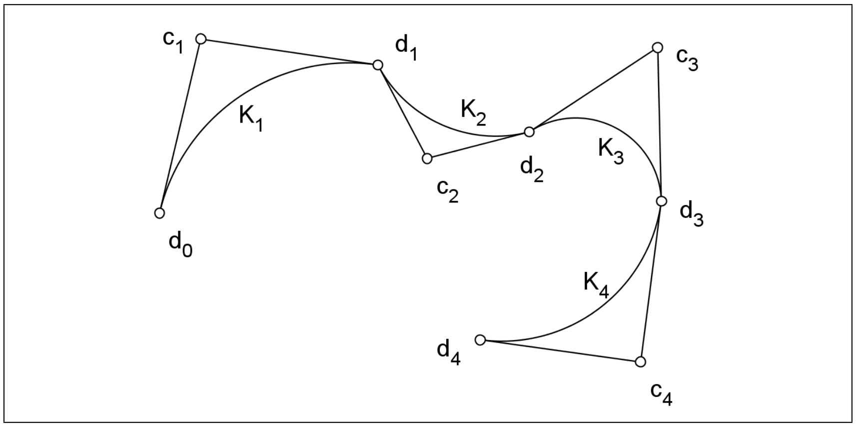

Figure 3: Connected circular arcs

We show that C can be represented as a rational B-spline curve of degree two. For this, we need an $n \in \mathbb{N}$, a knot vector T of the type

$$T : t_0 = t_1 = t_2 < t_3 \leq \ldots \leq t_n < t_{n+1} = t_{n+2} = t_{n+3}, \tag{2.4}$$

control points $\{\mathbf{b}_i\}_{i=0}^n$ in $\mathbb{R}^2$, and weights $\{w_i\}_{i=0}^n$ in $\mathbb{R}$ such that

$$C = \left\{ \mathbf{r}(t) = \frac{\sum_{i=0}^{n} w_i \mathbf{b}_i N_i(t)}{\sum_{i=0}^{n} w_i N_i(t)} : t \in [t_2, t_{n+1}] \right\}, \tag{2.5}$$

where $\{N_i\}_{i=0}^n$ denote the normalized B-splines of degree two with knot vector T (s. [2], [4]).

3 The Continuous Case

Let us assume in this section that there is no tangential transition from K_i to K_{i+1} in $\mathbf{d}_i$ for $i = 1, \ldots, L-1$. In order to show that C can be described as a NURBS $\mathbf{r}$ by (2.5), we set $n = 2L$, and choose a fixed knot vector of the type

$$T : t_0 = t_1 = t_2 < t_3 = t_4 < \ldots < t_{2L-1} = t_{2L} < t_{2L+1} = t_{2L+2} = t_{2L+3}, \tag{3.1}$$

i.e., $t_{2i-1} = t_{2i}$, $i = 2, \ldots, L$, and control points $\{\mathbf{b}_i\}_{i=0}^{2L}$ satisfying

$$\begin{aligned} \mathbf{b}_{2i} &= \mathbf{d}_i, & i &= 0, \ldots, L, \\ \mathbf{b}_{2i-1} &= \mathbf{c}_i, & i &= 1, \ldots, L. \end{aligned} \tag{3.2}$$

It is well known (s. e.g. [2], [4]) that the B-Splines $\{N_i\}_{i=0}^{2L}$ of degree two with knot vector T can be determined by the following recurrence relation:

$$N_0(t) = \begin{cases} \left(\dfrac{t_3 - t}{t_3 - t_2}\right)^2, & \text{if } t_2 \le t \le t_3 \\ 0, & \text{elsewhere,} \end{cases}$$

$$N_{2L}(t) = \begin{cases} 0, & \text{if } t_2 \le t \le t_{2L} \\ \left(\dfrac{t - t_{2L}}{t_{2L+1} - t_{2L}}\right)^2, & \text{elsewhere,} \end{cases}$$

$$N_{2i-1}(t) = \begin{cases} \dfrac{2}{(t_{2i+1} - t_{2i})^2}(t - t_{2i})(t_{2i+1} - t), & \text{if } t_{2i} \le t \le t_{2i+1} \\ 0, & \text{elsewhere} \end{cases}$$

for $i = 1, \ldots, L$, and

$$N_{2i}(t) = \begin{cases} \left(\dfrac{t - t_{2i}}{t_{2i+1} - t_{2i}}\right)^2, & \text{if } t_{2i} \le t \le t_{2i+1} \\ \left(\dfrac{t_{2i+3} - t}{t_{2i+3} - t_{2i+2}}\right)^2, & \text{if } t_{2i+2} < t \le t_{2i+3} \\ 0, & \text{elsewhere} \end{cases}$$

for $i = 1, \ldots, L-1$. In particular, we obtain for $\widetilde{t} \in [t_{2j}, t_{2j+1}]$, $j = 1, \ldots, L$:

$$\begin{aligned} \sum_{i=0}^{2L} N_i(\widetilde{t}) &= \sum_{i=2j-2}^{2j} N_i(\widetilde{t}) = 1, \\ N_i(\widetilde{t}) &= B_{i-2j+2}\left(\tfrac{\tilde{t} - t_{2j}}{t_{2j+1} - t_{2j}}\right), \quad i = 2j-2, 2j-1, 2j, \end{aligned} \tag{3.3}$$

where $\{B_i\}_{i=0}^{2}$ denote the Bernstein-polynomials of degree two. We define the weights $\{w_i\}_{i=0}^{2L}$ by

$$\begin{aligned} &w_{2i} = 1, \quad i = 0, \ldots, L, \\ &w_{2i-1} = \cos \sphericalangle(\mathbf{b}_{2i}, \mathbf{b}_{2i-2}, \mathbf{b}_{2i-1}), \quad i = 1, \ldots, L. \end{aligned} \tag{3.4}$$

Following (2.1) we easily verify that

$$K_i \subset \Delta(\mathbf{b}_{2i-2}, \mathbf{b}_{2i-1}, \mathbf{b}_{2i}), \quad i = 1, \ldots, L, \tag{3.5}$$

where $\Delta(\cdot)$ denotes the triangle with the specified vertices. In view of (2.2), it is easily verified that

$$0 < w_{2i-1} = \frac{\|\mathbf{b}_{2i} - \mathbf{b}_{2i-2}\|}{2\|\mathbf{b}_{2i-1} - \mathbf{b}_{2i-2}\|}, \ i = 1, \ldots, L.$$

Using the above defined parameters, we set

$$\mathbf{r}(t) = \frac{\sum_{i=0}^{2L} w_i \mathbf{b}_i N_i(t)}{\sum_{i=0}^{2L} w_i N_i(t)}, \quad t \in [t_2, t_{2L+1}], \tag{3.6}$$

and prove the following statements.

Theorem 3.1 *Assume that* $\mathbf{r}$ *is given as in (3.6). Then the following statements hold:*

(i) $C = \{\mathbf{r}(t) : t \in [t_2, t_{2L+1}]\}$;

(ii) *If* $\widetilde{\mathbf{r}}$ *is a NURBS defined by a knot vector* $\widetilde{T}$ *of the type (2.4), control points* $\{\widetilde{\mathbf{b}}_i\}_{i=0}^n$ *and weights* $\{\widetilde{w}_i\}_{i=0}^n$ *such that (2.5) holds, i.e.,*

$$C = \left\{ \widetilde{\mathbf{r}}(t) = \frac{\sum_{i=0}^{n} \widetilde{w}_i \widetilde{\mathbf{b}}_i \widetilde{N}_i(t)}{\sum_{i=0}^{n} \widetilde{w}_i \widetilde{N}_i(t)} : t \in [\widetilde{t}_2, \widetilde{t}_{n+1}] \right\}.$$

Then $n \geq 2L$.

Proof: (i) Suppose that $\mathbf{r}$ is defined by (3.6). We show that

$$K_j = \{\mathbf{r}(t) : t \in [t_{2j}, t_{2j+1}]\}$$

holds for $j = 1, \ldots, L$. Then, in view of

$$C = \bigcup_{j=1}^{L} K_j,$$

the statement follows immediately. Hence let a knot interval $[t_{2j}, t_{2j+1}]$ be given. Using (3.3), we obtain for $\widetilde{t} \in [t_{2j}, t_{2j+1}]$:

$$\begin{aligned} \mathbf{r}(\widetilde{t}) &= \frac{\sum_{i=0}^{2L} w_i \mathbf{b}_i N_i(\widetilde{t})}{\sum_{i=0}^{2L} w_i N_i(\widetilde{t})} = \frac{\sum_{i=2j-2}^{2j} w_i \mathbf{b}_i N_i(\widetilde{t})}{\sum_{i=2j-2}^{2j} w_i N_i(\widetilde{t})} \\ &= \frac{\sum_{i=2j-2}^{2j} w_i \mathbf{b}_i B_{i-2j+2}\left(\frac{\tilde{t}-t_{2j}}{t_{2j+1}-t_{2j}}\right)}{\sum_{i=2j-2}^{2j} w_i B_{i-2j+2}\left(\frac{\tilde{t}-t_{2j}}{t_{2j+1}-t_{2j}}\right)}. \end{aligned} \tag{3.7}$$

By (3.4), this implies that $\mathbf{r}|_{[t_{2j},t_{2j+1}]}$ is a rational Bézier-curve of degree two with $w_{2j-2} = w_{2j} = 1$, and

$$\begin{aligned} \mathbf{r}(t_{2j}) &= \frac{w_{2j-2}\mathbf{b}_{2j-2}}{w_{2j-2}} = \mathbf{b}_{2j-2} = \mathbf{d}_{j-1}, \\ \mathbf{r}(t_{2j+1}) &= \frac{w_{2j}\mathbf{b}_{2j}}{w_{2j}} = \mathbf{b}_{2j} = \mathbf{d}_j. \end{aligned}$$

In particular, this curve represents a conic section (s. [2]). Since by (3.5)

$$K_j \subset \Delta(\mathbf{b}_{2j-2}, \mathbf{b}_{2j-1}, \mathbf{b}_{2j}),$$

it follows that K_j is described by $\mathbf{r}|_{[t_{2j},t_{2j+1}]}$ if and only if w_{2j-1} is defined as in (3.4) [2], i.e., $0 < w_{2j-1} = \cos\sphericalangle(\mathbf{b}_{2j}, \mathbf{b}_{2j-2}, \mathbf{b}_{2j-1})$. Therefore, using this value for w_{2j-1}, we obtain:

$$K_j = \{\mathbf{r}(t) : t \in [t_{2j}, t_{2j+1}]\},$$

$j = 1, \ldots, L$.

(ii) For some $n \in \mathbb{N}$ let be given a knot vector $\widetilde{T}$ of the type (2.4), i.e.,

$$\widetilde{T} : \widetilde{t}_0 = \widetilde{t}_1 = \widetilde{t}_2 < \widetilde{t}_3 \leq \ldots \leq \widetilde{t}_n < \widetilde{t}_{n+1} = \widetilde{t}_{n+2} = \widetilde{t}_{n+3},$$

control points $\{\widetilde{\mathbf{b}}_i\}_{i=0}^n$ and weights $\{\widetilde{w}_i\}_{i=0}^n$ such that

$$C = \left\{ \widetilde{\mathbf{r}}(t) = \frac{\sum\limits_{i=0}^{n} \widetilde{w}_i \widetilde{\mathbf{b}}_i \widetilde{N}_i(t)}{\sum\limits_{i=0}^{n} \widetilde{w}_i \widetilde{N}_i(t)} : t \in [\widetilde{t}_2, \widetilde{t}_{n+1}] \right\}.$$

We show that the relation $n \geq 2L$ must follow. For the proof we consider arcs K_i and K_{i+1} for any $i \in \{1, \ldots, L-1\}$. Since we assume in this section that there is no tangential transition between both arcs in $\mathbf{d}_i$, $K_i \cup K_{i+1}$ does not form a continuously differentiable curve. Therefore, it cannot be described by a rational Bézier-curve. This implies that there exist knots $\widetilde{t}_{j_{i-1}} < \widetilde{t}_{j_i} < \widetilde{t}_{j_{i+1}}$ in $\widetilde{T}$ such that

$$\begin{aligned} K_i &= \{\widetilde{\mathbf{r}}(t) : t \in [\widetilde{t}_{j_{i-1}}, \widetilde{t}_{j_i}]\}, \\ K_{i+1} &= \{\widetilde{\mathbf{r}}(t) : t \in [\widetilde{t}_{j_i}, \widetilde{t}_{j_{i+1}}]\}, \end{aligned}$$

hence $\widetilde{\mathbf{r}}(\widetilde{t}_{j_i}) = \mathbf{d}_i$. It follows that $\widetilde{t}_{j_i}$ is a knot of multiplicity at least two. On the contrary, suppose that it is a simple knot, i.e., $\widetilde{t}_{j_i-1} < \widetilde{t}_{j_i} < \widetilde{t}_{j_i+1}$. Then each B-spline $\widetilde{N}_i$ and, therefore, the NURBS $\widetilde{\mathbf{r}}$ would be continuously differentiable in $\widetilde{t}_{j_i}$. But this contradicts the above arguments on $K_i \cup K_{i+1}$.

Hence it follows that $\widetilde{t}_{j_i}$ has multiplicity at least two for $i \in \{1, \ldots, L-1\}$. Thus there exist at least $L-1$ double knots in $(\widetilde{t}_2, \widetilde{t}_{n+1})$ which implies that

$$n - 2 \geq 2(L-1),$$

and, therefore, $n \geq 2L$. This proves statement (ii). □

Remark 3.2 Theorem 3.1 states that the curve $\mathbf{r}$ defined by (3.6) describes the given curve C by a minimal number of conditions.

4 The Smooth Case

Let us now assume that there is a tangential transition from K_i to K_{i+1} in $\mathbf{d}_i$, i.e., $\mathbf{c}_i, \mathbf{d}_i$ and $\mathbf{c}_{i+1}$ are collinear for $i = 1, \ldots, L-1$. Of course, we still assume that the assumptions (2.1) - (2.3) hold.

As in the preceding section we are looking for a minimal NURBS $\mathbf{r}$ describing C. As a first result we can apply Theorem 3.1 to the present case, setting $n = 2L$, and choosing knots, control points and weights as in (3.1), (3.2) and (3.4). Then from statement (i) in Theorem 3.1 we obtain immediately:

Theorem 4.1 *Assume that* $\mathbf{r}$ *is defined by (3.6). Then*

$$C = \{\mathbf{r}(t) : t \in [t_2, t_{2L+1}]\}.$$

The curve $\mathbf{r}$ is defined by $2L+1$ control points. But, as we will show later, for $L = 2$ the curve C can be described by a NURBS with a smaller number of control points. Hence in this case $\mathbf{r}$ fails to be minimal with respect to the number of parameters. Therefore, we are first interested in the question of how many parameters are necessary at least to describe C.

Lemma 4.2 *If* $\mathbf{r}$ *is a NURBS of type (2.5) which describes* C *then the relation*

$$n \geq L + 1$$

holds.

Proof: Since for each $i = 1, \ldots, L-1$ the points $\mathbf{c}_i, \mathbf{d}_i, \mathbf{c}_{i+1}$ are collinear, the $L+2$ points $\{\mathbf{d}_0, \mathbf{c}_1, \ldots, \mathbf{c}_L, \mathbf{d}_L\}$ form a polygon $\mathbf{P}$ with the property that $\mathbf{P}$ cannot be represented by a proper subset of these points. Therefore, since $\mathbf{P}$ can be taken as control polygon of $\mathbf{r}$, and since by the above arguments a control polygon with fewer points is impossible, the relation $n \geq L+1$ follows. □

In view of Lemma 4.2, we set

$$n = L + 1,$$

and define control points $\{\mathbf{b}_i\}_{i=0}^{L+1}$ by

$$\mathbf{b}_0 := \mathbf{d}_0, \quad \mathbf{b}_i := \mathbf{c}_i, \quad i = 1, \dots, L, \quad \mathbf{b}_{L+1} := \mathbf{d}_L.$$

Moreover, we need weights $\{w_i\}_{i=0}^{L+1}$ and knots $\{t_i\}_{i=0}^{L+4}$ of the type (2.4) such that the given curve C is described by a NURBS

$$\mathbf{r}(t) = \frac{\sum_{i=0}^{L+1} w_i \mathbf{b}_i N_i(t)}{\sum_{i=0}^{L+1} w_i N_i(t)}, \quad t \in [t_2, t_{L+2}]. \tag{4.1}$$

Note that such a curve $\mathbf{r}$ is minimal according to Lemma 4.2. Since by (2.3) $\mathbf{r}$ is not arbitrarily often differentiable in $\mathbf{d}_i$, $i = 1, \dots, L-1$, there must exist knots $t_{j_{i-1}} < t_{j_i} < t_{j_{i+1}}$ such that

$$K_i = \{\mathbf{r}(t) : t \in [t_{j_{i-1}}, t_{j_i}]\},$$
$$K_{i+1} = \{\mathbf{r}(t) : t \in [t_{j_i}, t_{j_{i+1}}]\}.$$

This implies $\mathbf{r}(t_{j_i}) = \mathbf{d}_i$ for $i = 1, \dots, L-1$. Moreover, since the knots satisfy the property

$$t_0 = t_1 = t_2 < t_3 \leq \dots \leq t_{L+1} < t_{L+2} = t_{L+3} = t_{L+4},$$

the relation

$$\mathbf{r}(t_{i+2}) = \mathbf{d}_i, \quad i = 1, \dots, L-1$$

must hold. Then the property $\mathbf{d}_i \neq \mathbf{d}_{i+1}$ for all i implies that

$$t_0 = t_1 = t_2 < t_3 < t_4 < \dots < t_{L+1} < t_{L+2} = t_{L+3} = t_{L+4}. \tag{4.2}$$

Thus, to solve our problem we have to find a knot vector T of the type (4.2) with simple knots $\{t_i\}_{i=3}^{L+1}$ and weights $\{w_i\}_{i=0}^{L+1}$ such that

$$C = \{\mathbf{r}(t) : t \in [t_2, t_{L+2}]\}, \tag{4.3}$$

where $\mathbf{r}$ is defined by (4.1). We may set

$$w_0 = 1 \text{ and } t_2 = 0.$$

Since the B-splines $\{N_i\}_{i=0}^{L+1}$ are invariant under a linear transformation of T, we may fix the knot t_{L+2}. Hence we still have to determine $2L$ parameters

$w_1, \ldots, w_{L+1}$, $t_3, \ldots, t_{L+1}$. In order to solve (4.3) these parameters must satisfy the following conditions:

$$\begin{aligned} &\mathbf{r}(t_{i+2}) = \mathbf{d}_i, \quad i = 1, \ldots, L-1, \\ &\mathbf{r}(\hat{t}_i) \in K_i \text{ for some } \hat{t}_i \in (t_{i+1}, t_{i+2}), \quad i = 1, \ldots, L. \end{aligned} \tag{4.4}$$

We will show in Theorem 4.3 that a curve $\mathbf{r}$ satisfying (4.4) describes the given curve C. Moreover, we will show in Remark 4.4 that (4.4) leads to a nonlinear system of $2L - 1$ equations for the $2L$ parameters

$$w_1, \ldots, w_{L+1}, \quad t_3 < t_4 < \ldots < t_{L+1}.$$

It turns out that we may also fix the knot t_3 such that

$$0 = t_2 < t_3 < t_{L+2}.$$

Theorem 4.3 *Assume that a NURBS* $\mathbf{r}$ *is given by (4.1) with knot vector* T *by (4.2) which satisfies the conditions (4.4). Then the following statements hold:*

(i) $K_i = \{\mathbf{r}(t) : t \in [t_{i+1}, t_{i+2}]\}, \quad i = 1, \ldots, L;$

(ii) $w_i > 0, \quad i = 0, \ldots, L+1;$

(iii)
$$w_{i+1} = w_i \frac{t_{i+3} - t_{i+2}}{t_{i+2} - t_{i+1}} \frac{\|\mathbf{d}_i - \mathbf{b}_i\|}{\|\mathbf{d}_i - \mathbf{b}_{i+1}\|} = \frac{t_{i+3} - t_{i+2}}{t_{i+2} - t_{i+1}} m_i w_i,$$

where
$$m_i := \frac{\|\mathbf{d}_i - \mathbf{b}_i\|}{\|\mathbf{d}_i - \mathbf{b}_{i+1}\|}, \quad i = 1, \ldots, L-1.$$

Proof: (i) Assume that $\mathbf{r}$ is given by (4.1) with knot vector T by (4.2) and let $i \in \{1, \ldots, L\}$. Since

$$N_j(t) = 0, \quad t \in I_i := [t_{i+1}, t_{i+2}], \quad j \neq i-1, i, i+1,$$

only the B-splines $\{N_j\}_{j=i-1}^{i+1}$ are relevant for I_i. These are determined by the well known recurrence relation (s. [2]) for $t \in I_i$ as follows:

$$\begin{aligned} N_{i-1}(t) &= \frac{(t_{i+2} - t)^2}{(t_{i+2} - t_{i+1})(t_{i+2} - t_i)}, \\ N_i(t) &= \frac{(t - t_i)(t_{i+2} - t)}{(t_{i+2} - t_{i+1})(t_{i+2} - t_i)} + \frac{(t_{i+3} - t)(t - t_{i+1})}{(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})}, \\ N_{i+1}(t) &= \frac{(t - t_{i+1})^2}{(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})}. \end{aligned}$$

In particular, we obtain for $t \in I_i$:

$$1 = \sum_{j=0}^{L+1} N_j(t) = \sum_{j=i-1}^{i+1} N_j(t).$$

Moreover, the Bernstein-polynomials $\{B_j\}_{j=0}^{2}$ related to I_i can be rewritten as

$$\begin{aligned} B_0(\widetilde{t}) &= \frac{t_{i+2}-t_i}{t_{i+2}-t_{i+1}} N_{i-1}(t), \\ B_1(\widetilde{t}) &= N_i(t) - \frac{t_{i+1}-t_i}{t_{i+2}-t_{i+1}} N_{i-1}(t) - \frac{t_{i+3}-t_{i+2}}{t_{i+2}-t_{i+1}} N_{i+1}(t), \\ B_2(\widetilde{t}) &= \frac{t_{i+3}-t_{i+1}}{t_{i+2}-t_{i+1}} N_{i+1}(t), \end{aligned} \tag{4.5}$$

where

$$t \in I_i, \quad \widetilde{t} = \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}} \in [0,1].$$

Since $t_1 = t_2$ resp. $t_{L+2} = t_{L+3}$, (4.5) implies for $i = 1$ resp. $i = L$:

$$B_0\left(\frac{t-t_2}{t_3-t_2}\right) = N_0(t) \text{ resp. } B_2\left(\frac{t-t_{L+1}}{t_{L+2}-t_{L+1}}\right) = N_{L+1}(t).$$

Since by assumption $\mathbf{r}$ satisfies the conditions (4.4), it follows

$$\mathbf{r}(t_{i+1}) = \mathbf{d}_{i-1},\ \mathbf{r}(t_{i+2}) = \mathbf{d}_i,\ \mathbf{r}(\hat{t}) \in K_i \ \text{ for some } \hat{t} \in (t_{i+1}, t_{i+2}). \tag{4.6}$$

Therefore, $\mathbf{r}$ has three common points with K_i.
We now show that $\mathbf{r}|_{I_i}$ actually forms a rational Bézier-curve of degree two with the control points $\{\mathbf{d}_{i-1}, \mathbf{c}_i, \mathbf{d}_i\}$ and positive weights. For

$$t \in I_i, \quad \widetilde{t} = \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}}$$

it follows from (4.5)

$$\begin{aligned} N_{i-1}(t) &= \frac{t_{i+2}-t_{i+1}}{t_{i+2}-t_i} B_0(\widetilde{t}), \\ N_{i+1}(t) &= \frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_{i+1}} B_2(\widetilde{t}), \\ N_i(t) &= B_1(\widetilde{t}) + \frac{t_{i+1}-t_i}{t_{i+2}-t_{i+1}} N_{i-1}(t) + \frac{t_{i+3}-t_{i+2}}{t_{i+2}-t_{i+1}} N_{i+1}(t) \\ &= B_1(\widetilde{t}) + \frac{t_{i+1}-t_i}{t_{i+2}-t_i} B_0(\widetilde{t}) + \frac{t_{i+3}-t_{i+2}}{t_{i+3}-t_{i+1}} B_2(\widetilde{t}). \end{aligned}$$

Setting

$$\begin{aligned} \widetilde{t}_0 &:= \frac{t_{i+2}-t_{i+1}}{t_{i+2}-t_i}, & \widetilde{t}_1 &:= \frac{t_{i+1}-t_i}{t_{i+2}-t_i}, \\ \widetilde{t}_2 &:= \frac{t_{i+3}-t_{i+2}}{t_{i+3}-t_{i+1}}, & \widetilde{t}_3 &:= \frac{t_{i+2}-t_{i+1}}{t_{i+3}-t_{i+1}}, \end{aligned} \tag{4.7}$$

we easily see that $0 < \widetilde{t}_j < 1$, $j = 0, \ldots, 3$, $\widetilde{t}_0 + \widetilde{t}_1 = 1$, $\widetilde{t}_2 + \widetilde{t}_3 = 1$.
Then, for the curve $\mathbf{r}$ satisfying (4.6), we obtain for any $t \in I_i$,

$$\begin{aligned} \mathbf{r}(t) &= \frac{\sum_{j=0}^{L+1} w_j \mathbf{b}_j N_j(t)}{\sum_{j=0}^{L+1} w_j N_j(t)} = \frac{\sum_{j=i-1}^{i+1} w_j \mathbf{b}_j N_j(t)}{\sum_{j=i-1}^{i+1} w_j N_j(t)}, \\ \sum_{j=i-1}^{i+1} w_j \mathbf{b}_j N_j(t) &= (w_{i-1}\mathbf{b}_{i-1}\widetilde{t}_0 + w_i \mathbf{b}_i \widetilde{t}_1)\, B_0(\widetilde{t}) + w_i \mathbf{b}_i B_1(\widetilde{t}) \\ &\quad + (w_i \mathbf{b}_i \widetilde{t}_2 + w_{i+1}\mathbf{b}_{i+1}\widetilde{t}_3)\, B_2(\widetilde{t}), \\ \sum_{j=i-1}^{i+1} w_j N_j(t) &= (w_{i-1}\widetilde{t}_0 + w_i \widetilde{t}_1)\, B_0(\widetilde{t}) + w_i B_1(\widetilde{t}) \\ &\quad + (w_i \widetilde{t}_2 + w_{i+1}\widetilde{t}_3)\, B_2(\widetilde{t}). \end{aligned} \tag{4.8}$$

Using the properties of the Bernstein-polynomials and (4.8) we obtain

$$\begin{aligned} \sum_{j=i-1}^{i+1} w_j \mathbf{b}_j N_j(t_{i+1}) &= w_{i-1}\mathbf{b}_{i-1}\widetilde{t}_0 + w_i \mathbf{b}_i \widetilde{t}_1, \\ \sum_{j=i-1}^{i+1} w_j \mathbf{b}_j N_j(t_{i+2}) &= w_i \mathbf{b}_i \widetilde{t}_2 + w_{i+1}\mathbf{b}_{i+1}\widetilde{t}_3. \end{aligned}$$

Therefore, by (4.6) we obtain the relations

$$\begin{aligned} \mathbf{d}_{i-1} &= \mathbf{r}(t_{i+1}) = \frac{w_{i-1}\mathbf{b}_{i-1}\widetilde{t}_0 + w_i \mathbf{b}_i \widetilde{t}_1}{w_{i-1}\widetilde{t}_0 + w_i \widetilde{t}_1}, \\ \mathbf{d}_i &= \mathbf{r}(t_{i+2}) = \frac{w_i \mathbf{b}_i \widetilde{t}_2 + w_{i+1}\mathbf{b}_{i+1}\widetilde{t}_3}{w_i \widetilde{t}_2 + w_{i+1}\widetilde{t}_3}. \end{aligned} \tag{4.9}$$

Setting

$$\begin{aligned} \widetilde{w}_{i-1} &:= w_{i-1}\widetilde{t}_0 + w_i \widetilde{t}_1, \quad \widetilde{w}_i := w_i \\ \widetilde{w}_{i+1} &:= w_i \widetilde{t}_2 + w_{i+1}\widetilde{t}_3, \\ \widetilde{\mathbf{b}}_0 &:= \mathbf{d}_{i-1},\ \widetilde{\mathbf{b}}_1 := \mathbf{b}_i,\ \widetilde{\mathbf{b}}_2 := \mathbf{d}_i \end{aligned}$$

we can rewrite (4.9) to

$$\begin{aligned} w_{i-1}\mathbf{b}_{i-1}\widetilde{t}_0 + w_i \mathbf{b}_i \widetilde{t}_1 &= \widetilde{w}_{i-1}\widetilde{\mathbf{b}}_0, \\ w_i \mathbf{b}_i \widetilde{t}_2 + w_{i+1}\mathbf{b}_{i+1}\widetilde{t}_3 &= \widetilde{w}_{i+1}\widetilde{\mathbf{b}}_2. \end{aligned}$$

Hence,(4.8) can be reformulated for any $t \in I_i$ as

$$\mathbf{r}(t) = \frac{\widetilde{w}_{i-1}\widetilde{\mathbf{b}}_0 B_0(\widetilde{t}) + \widetilde{w}_i \widetilde{\mathbf{b}}_1 B_1(\widetilde{t}) + \widetilde{w}_{i+1}\widetilde{\mathbf{b}}_2 B_2(\widetilde{t})}{\widetilde{w}_{i-1} B_0(\widetilde{t}) + \widetilde{w}_i B_1(\widetilde{t}) + \widetilde{w}_{i+1} B_2(\widetilde{t})}. \tag{4.10}$$

This implies that $\mathbf{r}|_{I_i}$ is a rational Bézier-curve of degree two with the control points $\{\widetilde{\mathbf{b}}_0, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2\}$ and the weights $\{\widetilde{w}_{i-1}, \widetilde{w}_i, \widetilde{w}_{i+1}\}$.
In the proof of statement (ii) we will show that all weights $\{w_0, \dots, w_{L+1}\}$ are positive. Using the formulas (4.7) we then can easily conclude that the weights $\{\widetilde{w}_0, \dots, \widetilde{w}_{L+1}\}$ have the same property.
We now determine the derivatives $\frac{d}{dt}\mathbf{r}$ in $t = t_{i+1}$ and $t = t_{i+2}$. By (4.10) it is easily seen (s. [2]) that

$$\begin{aligned} \frac{d}{dt}\mathbf{r}(t_{i+1}) &= \frac{2\widetilde{w}_i}{\widetilde{w}_{i-1}}(\widetilde{\mathbf{b}}_1 - \widetilde{\mathbf{b}}_0), \\ \frac{d}{dt}\mathbf{r}(t_{i+2}) &= \frac{2\widetilde{w}_i}{\widetilde{w}_{i+1}}(\widetilde{\mathbf{b}}_1 - \widetilde{\mathbf{b}}_2). \end{aligned} \tag{4.11}$$

Since $\widetilde{\mathbf{b}}_1 - \widetilde{\mathbf{b}}_0 = \mathbf{c}_i - \mathbf{d}_{i-1}$ and $\widetilde{\mathbf{b}}_1 - \widetilde{\mathbf{b}}_2 = \mathbf{c}_i - \mathbf{d}_i$, by hypothesis both vectors have the same directions as the tangents at K_i in $\mathbf{d}_{i-1}$ resp. $\mathbf{d}_i$. Therefore, the curve $\mathbf{r}$ lies tangential at K_i in $\mathbf{d}_{i-1}$ and $\mathbf{d}_i$. Hence it follows from (4.4) that

$$K_i = \{\mathbf{r}(t) : t \in I_i\}.$$

This proves statement (i).

(ii) Now we show that $w_i > 0$ holds for $i = 0, \dots, L+1$. First it follows from (2.1) for $i = 1, \dots, L$ that

$$K_i \subset \mathrm{conv}(\mathbf{d}_{i-1}, \mathbf{c}_i, \mathbf{d}_i)$$

In particular, since the points $\{\mathbf{b}_{i-1}, \mathbf{d}_{i-1}, \mathbf{b}_i\}$, $i = 2, \dots, L$ are collinear, it follows for $i = 1, \dots, L$ that

$$K_i \subset \mathrm{conv}(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{b}_{i+1}).$$

Hence we have for $t \in I_i$,

$$\mathbf{r}(t) \in K_i \subset \mathrm{conv}(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{b}_{i+1}).$$

Moreover, by (4.8) we obtain for $t \in I_i$ that

$$\mathbf{r}(t) = \sum_{j=i-1}^{i+1} \frac{w_j N_j(t)}{\sum\limits_{j=i-1}^{i+1} w_j N_j(t)} \mathbf{b}_j = \sum_{j=i-1}^{i+1} \lambda_{i,j} \mathbf{b}_j,$$

where

$$\lambda_{i,j} = \frac{w_j N_j(t)}{\sum_{j=i-1}^{i+1} w_j N_j(t)}, \quad j = i-1, i, i+1.$$

Thus we have got for $i = 1, \ldots, L$ that

$$\lambda_{i,j} \geq 0, \quad \sum_{j=i-1}^{i+1} \lambda_{i,j} = 1.$$

Then, since $w_0 = 1$, $\lambda_{1,0} \geq 0$ and $N_j(t) \geq 0$ for all j and all t, for $i = 1$ we obtain

$$\sum_{j=0}^{2} w_j N_j(t) > 0.$$

Therefore, from this and the relations $\lambda_{1,j} \geq 0$ for $j = 1, 2$ it follows that $w_j \geq 0$ for $j = 1, 2$. However, $w_1 = 0$ is not possible, because otherwise the curve $\mathbf{r}|_{I_1}$ would represent the straight line $[\mathbf{d}_0, \mathbf{d}_1]$. Analogously, $w_2 = 0$ is not possible. This proves $w_j > 0$, $j = 1, 2$.
Using analogous arguments for the intervals $I_2, \ldots, I_L$ we can conclude that also the weights $w_3, \ldots, w_{L+1}$ are positive.
This proves statement (ii).

(iii) Rewriting (4.9) and using (4.7) we obtain for $i = 2, \ldots, L$ the relations

$$w_{i-1}\widetilde{t}_0(\mathbf{d}_{i-1} - \mathbf{b}_{i-1}) = w_i\widetilde{t}_1(\mathbf{b}_i - \mathbf{d}_{i-1})$$

and

$$w_{i-1}(t_{i+2} - t_{i+1})(\mathbf{d}_{i-1} - \mathbf{b}_{i-1}) = w_i(t_{i+1} - t_i)(\mathbf{b}_i - \mathbf{d}_{i-1}). \tag{4.12}$$

Then, since by statement (ii) every w_i is positive, from (4.12) it follows for $i = 1, \ldots, L-1$ that

$$w_{i+1} = w_i \, \frac{t_{i+3} - t_{i+2}}{t_{i+2} - t_{i+1}} \, \frac{\|\mathbf{d}_i - \mathbf{b}_i\|}{\|\mathbf{d}_i - \mathbf{b}_{i+1}\|} = \frac{t_{i+3} - t_{i+2}}{t_{i+2} - t_{i+1}} m_i w_i. \tag{4.13}$$

This proves statement (iii). □

Remark 4.4 In order to determine the curve $\mathbf{r}$ considered in Theorem 4.3 we have first to solve the $L-1$ equations (4.13). In addition, we need further L conditions. These, in view of (4.4), can be derived from the relations

$$\mathbf{r}(\hat{t}_i) \in K_i \ \text{ for some } \hat{t}_i \in (t_{i+1}, t_{i+2}), \ i = 1, \ldots, L. \tag{4.14}$$

If we find solutions

$$w_1, \ldots, w_{L+1}, \ t_3 < t_4 < \ldots < t_{L+1} < t_{L+2}$$

of (4.13) and (4.14), we have verified the existence of $\mathbf{r}$ (dependent on the fixed chosen parameters $0 < t_3 < t_{L+2}$).
It turns out that our arguments in the following can be considerably simplified replacing the L conditions in (4.14) by equivalent conditions which are related to the Bézier-representation (4.10) of $\mathbf{r}$ in the interval I_i. In this case the weight w_i can be defined by a method already used in Section 3 (s. [2]).
Assume that for some $i \in \{1, \ldots, L\}$ the representation (4.10) is given, i.e.,

$$\mathbf{r}(t) = \frac{\widetilde{w}_{i-1}\widetilde{\mathbf{b}}_0 B_0(\widetilde{t}) + \widetilde{w}_i \widetilde{\mathbf{b}}_1 B_1(\widetilde{t}) + \widetilde{w}_{i+1}\widetilde{\mathbf{b}}_2 B_2(\widetilde{t})}{\widetilde{w}_{i-1} B_0(\widetilde{t}) + \widetilde{w}_i B_1(\widetilde{t}) + \widetilde{w}_{i+1} B_2(\widetilde{t})}, \quad t \in I_i.$$

First we set

$$\overline{w}_j := \frac{\widetilde{w}_j}{\widetilde{w}_{i-1}}, \quad j = i-1, i, i+1,$$
$$\rho := \sqrt{\frac{\overline{w}_{i-1}}{\overline{w}_{i+1}}} = \sqrt{\frac{1}{\overline{w}_{i+1}}} = \sqrt{\frac{\widetilde{w}_{i-1}}{\widetilde{w}_{i+1}}}$$

and

$$\hat{w}_j := \rho^j \overline{w}_j, \quad j = i-1, i, i+1. \tag{4.15}$$

This implies

$$\begin{aligned} &\hat{w}_{i-1} = \overline{w}_{i-1} = 1, \quad \hat{w}_{i+1} = \rho^2 \overline{w}_{i+1} = 1, \\ &\hat{w}_i = \rho \overline{w}_i = \sqrt{\frac{1}{\overline{w}_{i+1}}}\,\overline{w}_i = \sqrt{\frac{\tilde{w}_{i-1}}{\tilde{w}_{i+1}}}\frac{\tilde{w}_i}{\tilde{w}_{i-1}} = \frac{\tilde{w}_i}{\sqrt{\tilde{w}_{i-1}\tilde{w}_{i+1}}}. \end{aligned} \tag{4.16}$$

Now setting

$$\hat{w}_i = \cos \beta_i > 0, \tag{4.17}$$

where $\beta_i = \sphericalangle(\widetilde{\mathbf{b}}_2, \widetilde{\mathbf{b}}_0, \widetilde{\mathbf{b}}_1) = \sphericalangle(\mathbf{d}_i, \mathbf{d}_{i-1}, \mathbf{b}_i)$, we use the well known property (s. [2]) that the Bézier-curve

$$\mathbf{r}(t) = \frac{\hat{w}_{i-1}\widetilde{\mathbf{b}}_0 B_0(\hat{t}) + \hat{w}_i \widetilde{\mathbf{b}}_1 B_1(\hat{t}) + \hat{w}_{i+1}\widetilde{\mathbf{b}}_2 B_2(\hat{t})}{\hat{w}_{i-1} B_0(\hat{t}) + \hat{w}_i B_1(\hat{t}) + \hat{w}_{i+1} B_2(\hat{t})}, \quad t \in I_i,\ \hat{t} \in [0,1]$$

describes the circular arc K_i with the weights defined in (4.15) and (4.17).
Therefore, using the conditions (4.13), (4.15) and (4.16) we develop a system of equations for the weights $\{w_1, \ldots, w_{L+1}\}$ and knots $\{t_4 < \ldots < t_{L+1}\}$ as follows. Applying (4.7) and (4.9) we first obtain

$$\begin{aligned} \widetilde{w}_{i-1} &= \frac{1}{t_{i+2} - t_i}[(t_{i+2} - t_{i+1})w_{i-1} + (t_{i+1} - t_i)w_i], \\ \widetilde{w}_i &= w_i, \\ \widetilde{w}_{i+1} &= \frac{1}{t_{i+3} - t_{i+1}}[(t_{i+3} - t_{i+2})w_i + (t_{i+2} - t_{i+1})w_{i+1}]. \end{aligned} \tag{4.18}$$

To eliminate the weights w_{i-1} and w_{i+1}, using (4.13) we rewrite (4.18) for $i \in \{2, \dots, L-1\}$ as

$$\begin{aligned}\widetilde{w}_{i-1} &= \frac{t_{i+1}-t_i}{t_{i+2}-t_i}(1+\frac{1}{m_{i-1}})w_i,\\ \widetilde{w}_{i+1} &= \frac{t_{i+3}-t_{i+2}}{t_{i+3}-t_{i+1}}(1+m_i)w_i.\end{aligned} \tag{4.19}$$

This together with (4.16) implies

$$\begin{aligned}\hat{w}_i &= \frac{w_i\sqrt{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})m_{i-1}}}{\sqrt{w_i^2(t_{i+1}-t_i)(t_{i+3}-t_{i+2})(1+m_i)(1+m_{i-1})}}\\ &= \frac{\sqrt{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})m_{i-1}}}{\sqrt{(t_{i+1}-t_i)(t_{i+3}-t_{i+2})(1+m_i)(1+m_{i-1})}}, \quad i = 2, \dots, L-1.\end{aligned} \tag{4.20}$$

Moreover, considering (4.17) in (4.20), for $i = 2, \dots, L-1$ we obtain

$$\frac{(t_{i+1}-t_i)(t_{i+3}-t_{i+2})}{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})} = \frac{m_{i-1}}{(1+m_{i-1})(1+m_i)\cos^2\beta_i} =: n_i. \tag{4.21}$$

Since $m_i > 0$ holds for each i, it follows that also $n_i > 0$, $i = 2, \dots, L-1$.
A special situation is given for the cases $i = 1$ and $i = L$. Assume first that $i = 1$. Then, since $t_1 = t_2 = 0$, it follows from (4.13) and (4.18) that

$$\begin{aligned}&\widetilde{w}_0 = w_0 = 1,\ \widetilde{w}_1 = w_1,\\ &\widetilde{w}_2 = \tfrac{1}{t_4}[(t_4-t_3)w_1 + t_3w_2] = \tfrac{1}{t_4}(t_4-t_3)(1+m_1)w_1.\end{aligned}$$

Moreover, from (4.16) and (4.17) it follows that

$$\cos\beta_1 = \frac{w_1}{\sqrt{\widetilde{w}_2}} = \frac{w_1}{\sqrt{\frac{1}{t_4}(t_4-t_3)(1+m_1)w_1}}.$$

This finally implies that

$$w_1 = \frac{(t_4-t_3)(1+m_1)}{t_4}\cos^2\beta_1 = \frac{(t_4-t_3)(1+m_1)}{t_4-t_2}\cos^2\beta_1 w_0. \tag{4.22}$$

Assume now that $i = L$. Then, since $t_{L+2} = t_{L+3}$, it follows from (4.13) and (4.18) that

$$\begin{aligned}\widetilde{w}_{L-1} &= \frac{1}{t_{L+2}-t_L}[(t_{L+2}-t_{L+1})w_{L-1} + (t_{L+1}-t_L)w_L]\\ &= \frac{1}{t_{L+2}-t_L}(t_{L+1}-t_L)(1+\frac{1}{m_{L-1}})w_L,\\ \widetilde{w}_L &= w_L, \quad \widetilde{w}_{L+1} = w_{L+1}.\end{aligned}$$

Moreover, from (4.16) and (4.17) it follows that

$$\cos\beta_L = \hat{w}_L = \frac{w_L\sqrt{(t_{L+2}-t_L)m_{L-1}}}{\sqrt{(t_{L+1}-t_L)(1+m_{L-1})w_L w_{L+1}}}.$$

This finally implies that

$$w_{L+1} = w_L \frac{(t_{L+2}-t_L)m_{L-1}}{(t_{L+1}-t_L)(1+m_{L-1})\cos^2\beta_L}. \tag{4.23}$$

Summary The unknown weights $\{w_1,\dots,w_{L+1}\}$ and knots $\{t_4 < \dots < t_{L+1}\}$ have necessarily to satisfy the $2L-1$ equations (4.13), (4.21), (4.22) and (4.23). If solutions exist, these parameters define a NURBS $\mathbf{r}$ by (4.1) which describes the given curve C by a minimal number of conditions.

First it turns out that in the case $L = 2$ a solution always exists.

The case $L = 2$**:** We need a knot vector of the type (4.2), i.e.,

$$t_0 = t_1 = t_2 = 0 < t_3 < t_4 = t_5 = t_6$$

with $0 < t_3 < t_4$ fixed numbers. Therefore, using (4.13), (4.22) and (4.23) we have only to determine the weights $\{w_1, w_2, w_3\}$ ($w_0 = 1$ by hypothesis). We then obtain

$$\begin{aligned}
w_1 &= \frac{t_4-t_3}{t_4}(1+m_1)\cos^2\beta_1 > 0,\\
w_2 &= \frac{(t_4-t_3)m_1}{t_3}w_1 > 0,\\
w_3 &= \frac{t_4 m_1}{t_3(1+m_1)\cos^2\beta_2}w_2 > 0.
\end{aligned} \tag{4.24}$$

This implies that a NURBS $\mathbf{r}$ having the weights from (4.24) describes $C = K_1 \cup K_2$.

It turns out that in the case $L > 2$ a corresponding minimal NURBS does not always exist. However, in the following section we are able to give a characterization of existence.

5 Characterization of Existence of Minimal NURBS

Let us assume that $L > 2$. Moreover, assume that all arguments in Section 4 still hold. Hence we have to determine a set of knots (in case, they exist)

$$t_0 = t_1 = t_2 < t_3 < \dots < t_{L+1} < t_{L+2} = t_{L+3} = t_{L+4}, \tag{5.1}$$

where we assume that $0 < t_3 < t_{L+2}$ are fixed numbers. We use the numbers defined in (4.21), i.e.,

$$0 < n_i := \frac{1}{(1+1/m_{i-1})(1+m_i)\cos^2\beta_i}, \qquad i = 2,\dots,L-1,$$

and define

$$\gamma_{L+1} = \gamma_L = 1,$$

and recursively

$$\gamma_i = \gamma_{i+1} - n_i\gamma_{i+2}, \qquad i = L-1,\dots,2.$$

Then we get a nice necessary and sufficient characterization.

Theorem 5.1 *A minimal NURBS of the type (4.1) with knot vector (5.1) describing the given set C of L circular arcs exists if and only if*

$$\gamma_i > 0, \qquad i = 2,\dots,L-1. \tag{5.2}$$

Proof: We define

$$d_i = t_{i+1} - t_i, \qquad i = 2,\dots,L+1$$

and

$$\lambda_i = \frac{d_{i+1}}{d_i}, \qquad i = 2,\dots,L.$$

We rewrite the conditions (4.13), (4.22) and (4.23) for the weights as

$$w_1 = w_0\frac{\lambda_2}{1+\lambda_2}(1+m_1)\cos^2\beta_1,$$

and

$$w_{i+1} = w_i\lambda_{i+1}m_i, \qquad i = 1,\dots,L-1,$$

and

$$w_{L+1} = w_L(1+\lambda_L)\frac{1}{(1+1/m_{L-1})\cos^2\beta_L}.$$

Moreover, we have to satisfy the conditions (4.21), i.e.,

$$\frac{1}{(1+\lambda_i)(1+1/\lambda_{i+1})} = n_i, \qquad i = 2,\dots,L-1. \tag{5.3}$$

First, assume there is a set of knots and weights describing the set C of L circular arcs. Then all these conditions have to be fulfilled. We need to show that $\gamma_i > 0$ for all i. We have

$$\lambda_i = \frac{1}{(1+1/\lambda_{i+1})n_i} - 1, \qquad i = 2,\dots,L-1.$$

Since $\lambda_L > 0$, we have

$$\lambda_{L-1} < \frac{1}{n_{L-1}} - 1 = \frac{\gamma_L - n_{L-1}\gamma_{L+1}}{n_{L-1}\gamma_{L+1}} = \frac{\gamma_{L-1}}{n_{L-1}\gamma_{L+1}} = \frac{\gamma_{L-1}}{\gamma_L - \gamma_{L-1}}.$$

With $\lambda_{L-1} > 0$ we get $\gamma_{L-1} > 0$. We now show recursively that

$$\gamma_{i+1} > \gamma_i > 0$$

and

$$\lambda_i < \frac{\gamma_i}{\gamma_{i+1} - \gamma_i}, \qquad i = L-1, \ldots, 2.$$

This holds for $i = L - 1$. Assume it is true for i. Then

$$\lambda_{i-1} < \frac{1}{\left(1 + \dfrac{\gamma_{i+1} - \gamma_i}{\gamma_i}\right) n_{i-1}} - 1 = \frac{\gamma_i - \gamma_{i+1} n_{i-1}}{\gamma_{i+1} n_{i-1}} = \frac{\gamma_{i-1}}{\gamma_i - \gamma_{i-1}}.$$

Using the definition of γ_{i-1} and the induction hypothesis we get

$$\gamma_i - \gamma_{i-1} = n_{i-1}\gamma_{i+1} > 0.$$

Moreover, since $\lambda_{i-1} > 0$, we have $\gamma_{i-1} > 0$.

To prove the converse we assume that $\gamma_i > 0$ for all i. It suffices to show that there are

$$\lambda_i > 0, \qquad i = 2, \ldots, L$$

satisfying (5.3). Then we can construct the knots t_i as in (5.1), and the weights $w_i > 0$ for all i as in (4.13), (4.22) and (4.23) such that the NURBS $\mathbf{r}$ defined by (4.1) represents the set C of L circular arcs. Since all $\gamma_i > 0$, we get by the definition of γ_i

$$0 < \gamma_2 < \ldots < \gamma_L = 1.$$

We choose any λ_2 with

$$0 < \lambda_2 < \frac{\gamma_2}{\gamma_3 - \gamma_2},$$

and define

$$\lambda_{i+1} = \frac{1}{\dfrac{1}{(1 + \lambda_i) n_i} - 1} \qquad i = 2, \ldots, L-1.$$

Then these λ_i satisfy (5.3). We show recursively

$$0 < \lambda_i < \frac{\gamma_i}{\gamma_{i+1} - \gamma_i}, \qquad i = 2, \ldots, L.$$

Assume this holds for some $i \le L-1$. Then

$$(1+\lambda_i)n_i < \left(1 + \frac{\gamma_i}{\gamma_{i+1}-\gamma_i}\right) n_i = \frac{\gamma_{i+1}}{\gamma_{i+1}-\gamma_i} n_i = \frac{\gamma_{i+1}}{\gamma_{i+2}} \le 1.$$

Thus

$$\lambda_{i+1} > 0.$$

Moreover,

$$\lambda_{i+1} < \frac{1}{\dfrac{1}{\left(1 + \dfrac{\gamma_i}{\gamma_{i+1}-\gamma_i}\right) n_i} - 1} = \frac{\gamma_{i+1}}{\gamma_{i+2}-\gamma_{i+1}}.$$

This completes the proof of the Theorem. □

For reference, the conditions of Theorem 5.1 are

$$\begin{aligned}
\gamma_{L-1} &= 1 - n_{L-1} > 0, \\
\gamma_{L-2} &= 1 - n_{L-1} - n_{L-2} > 0 \\
\gamma_{L-3} &= 1 - n_{L-1} - n_{L-2} - n_{L-3} + n_{L-3} n_{L-1} > 0
\end{aligned}$$

and so on. It is not true that $\gamma_i > 0$ always implies $\gamma_{i+1} > 0$ even if $n_i < 1$ for all i. On the other hand, the conditions imply

$$0 < \gamma_2 < \ldots < \gamma_L = 1$$

and thus

$$n_i = \frac{\gamma_{i+1}-\gamma_i}{\gamma_{i+2}} < 1, \qquad i = 2, \ldots, L-1.$$

But this is only a necessary condition.

Example 5.2 *Assume that*

$$0 < n_2 = \ldots = n_{L-1} = c < 1.$$

Then we have to study the two term recursion

$$s_0 = s_1 = 1, \quad s_{i+1} = s_i - c s_{i-1}.$$

For $c < 1/4$, this recursion has the solution

$$s_i = \alpha_1 \mu_1^i + \alpha_2 \mu_2^i, \qquad i \in \mathbb{N}_0$$

where

$$\mu_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}, \quad \alpha_1 = \frac{1-\mu_2}{\mu_1-\mu_2}, \quad \alpha_2 = \frac{\mu_1 - 1}{\mu_1 - \mu_2}.$$

So we have $s_i > 0$ *for all* i *in this case. For* $c = 1/4$*, the recursion has the solution*

$$s_i = \frac{1+i}{2^i}.$$

Again, the condition of Theorem 5.1 is fulfilled for all L*. For* $c > 1/4$ *however, the condition fails. This example can be applied to a logarithmic volute, consisting of arcs with the same angle* α*, where the radii of the arcs satisfy*

$$r_{i+1} = dr_i, \qquad i = 1, \ldots, L.$$

Obviously we obtain

$$c = \frac{1}{(1+1/m_{i-1})(1+m_i)\cos^2\gamma_i} = \frac{1}{(1+1/d)(1+d)\cos^2\alpha/2}.$$

For a logarithmic volute consisting of quarter circles, we need

$$d > 3 + \sqrt{8}.$$

Otherwise, only a few arcs can be handled. E.g., for the circular case $d = 1$*,* $L = 3$ *is the limit, and for* $d = 1/2$*, it is* $L = 4$*.*

Now we apply Theorem 5.1 to some small numbers L and determine the needed weights and knots. Recall that $w_0 = 1$ and $0 < t_3 < t_{L+2}$ are fixed numbers.

The case $L = 3$**:** By Theorem 5.1 a minimal NURBS exists if and only if $\gamma_2 = 1 - n_2 > 0$, or equivalently,

$$n_2 < 1. \tag{5.4}$$

If this holds, by (4.13), (4.22) and (4.23) we obtain the needed weights as

$$\begin{aligned} w_1 &= \frac{(t_4 - t_3)(1+m_1)}{t_4}\cos^2\beta_1, \\ w_{i+1} &= \frac{t_{i+3} - t_{i+2}}{t_{i+2} - t_{i+1}} m_i w_i, \quad i = 1, 2, \\ w_4 &= \frac{(t_5 - t_3)m_2}{(t_4 - t_3)(1+m_2)\cos^2\beta_3} w_3. \end{aligned} \tag{5.5}$$

Moreover, by (4.21) we can easily verify that

$$t_4 = \frac{t_3 t_5}{t_3 + n_2(t_5 - t_3)}. \tag{5.6}$$

It then follows from Theorem 5.1 that the parameters given in (5.5) and (5.6) define a NURBS $\mathbf{r}$ describing $C = K_1 \cup K_2 \cup K_3$.

The case $L = 4$: By Theorem 5.1 a minimal NURBS exists if and only if $\gamma_3 = 1 - n_3 > 0$ and $\gamma_2 = 1 - n_3 - n_2 > 0$, or equivalently,

$$n_2 + n_3 < 1. \tag{5.7}$$

If this holds, by (4.13), (4.22) and (4.23) we obtain the needed weights as

$$\begin{aligned} &w_i \quad \text{as in (5.5)}, \quad i = 1, 2, 3, \\ &w_4 = \frac{t_6 - t_5}{t_5 - t_4} m_3 w_3, \\ &w_5 = \frac{(t_6 - t_4) m_3}{(t_5 - t_4)(1 + m_3) \cos^2 \beta_4} w_4. \end{aligned} \tag{5.8}$$

Moreover, by (4.21) we can easily verify that

$$t_4 = \frac{(1 - n_3) t_3 t_6}{(1 - n_3) t_3 + n_2 (t_6 - t_3)}, \quad t_5 = \frac{(1 - n_2)(1 - n_3) t_3 t_6}{(1 - n_2 - n_3) t_3 + n_2 n_3 t_6}. \tag{5.9}$$

It then follows from Theorem 5.1 that the parameters given in (5.8) and (5.9) define a NURBS $\mathbf{r}$ describing $C = \cup_{i=1}^4 K_i$.

The case $L = 5$: By Theorem 5.1 a minimal NURBS exists if and only if $\gamma_4 = 1 - n_4 > 0$, $\gamma_3 = 1 - n_4 - n_3 > 0$ and $\gamma_2 = 1 - n_4 - n_3 - n_2(1 - n_4) > 0$. These conditions are obviously equivalent to

$$1 - n_4 - n_3 - n_2(1 - n_4) > 0, \quad n_i < 1, \quad i = 2, 3, 4. \tag{5.10}$$

If these relations hold, by (4.13), (4.22) and (4.23) we obtain the needed weights as

$$\begin{aligned} &w_i \quad \text{as in (5.8)}, \quad i = 1, \ldots, 4, \\ &w_5 = \frac{t_7 - t_6}{t_6 - t_5} m_4 w_4, \\ &w_6 = \frac{(t_7 - t_5) m_4}{(t_6 - t_5)(1 + m_4) \cos^2 \beta_5} w_5. \end{aligned} \tag{5.11}$$

Moreover, by (4.21) we can easily verify that

$$\begin{aligned} t_4 &= \frac{(1 - n_3 - n_4) t_3 t_7}{(1 - n_3 - n_4 - n_2(1 - n_4)) t_3 + n_2 (1 - n_4) t_7}, \\ t_5 &= \frac{(1 - n_2)(1 - n_3 - n_4) t_3 t_7}{(1 - n_3 - n_4 - n_2(1 - n_4)) t_3 + n_2 n_3 t_7}, \\ t_6 &= \frac{(1 - n_2 - n_3)(1 - n_3 - n_4) t_3 t_7}{(1 - n_3)(1 - n_3 - n_4 - n_2(1 - n_4)) t_3 + n_2 n_3 n_4 t_7}. \end{aligned} \tag{5.12}$$

It then follows from Theorem 5.1 that the parameters given in (5.11) and (5.12) define a NURBS $\mathbf{r}$ describing $C = \cup_{i=1}^5 K_i$.

Example 5.3 *We construct a NURBS* $\mathbf{r}$ *consisting of* 4 *circular arcs with decreasing radii. This corresponds to the first rotation of a volute at Ionic pillars (s. [1]). As the centres of the circular arcs we take the points*

$$\mathbf{p}_1 = [0,0],\ \mathbf{p}_2 = [1,0],\ \mathbf{p}_3 = [1,-1],\ \mathbf{p}_4 = [0,-1].$$

Moreover, as control points of $\mathbf{r}$ *we choose the points*

$$\begin{array}{l}\mathbf{b}_0 = [0,10],\ \mathbf{b}_1 = [10,10],\ \mathbf{b}_2 = [10,-9],\\ \mathbf{b}_3 = [-7,-9],\ \mathbf{b}_4 = [-7,6],\ \mathbf{b}_5 = [0,6].\end{array}$$

Using (4.21) we easily calculate

$$n_2 = 0.49536,\ n_3 = 0.49412$$

which implies that

$$n_2 + n_3 < 1.$$

Therefore, by (5.7) there exists a solution of our problem depending on t_3 *and* t_6. *Setting* $t_3 = 1$, $t_6 = 4$ *and using (5.8) and (5.9) we get the weights and the knots of the minimal NURBS* $\mathbf{r}$ *as*

$$\begin{array}{l}w_0 = 1,\ w_1 = 0.16473 \cdot 10^{-1},\ \ w_2 = 0.29017 \cdot 10^{-3},\\ w_3 = 0.33065 \cdot 10^{-3},\ w_4 = 0.69847 \cdot 10^{-1},\ w_5 = 13.84532\end{array}$$

resp.

$$t_0 = t_1 = t_2 = 0,\ t_3 = 1,\ t_4 = 1.01585,\ t_5 = 1.03191,\ t_6 = t_7 = t_8 = 4$$

(all values are rounded). Then defining $\mathbf{r}$ *by (4.1) we obtain the NURBS which describes* $C = \cup_{i=1}^4 K_i$ *(s. Fig. 4).*

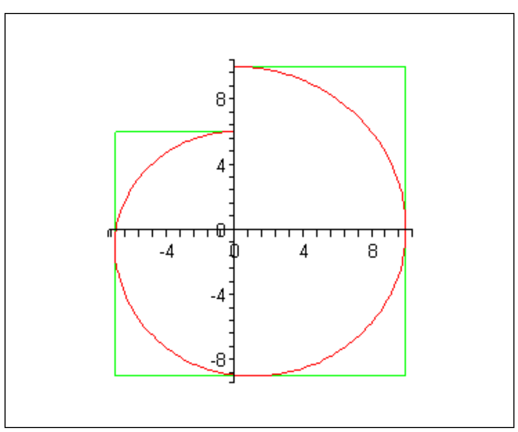

Figure 4: NURBS and control polygon: Example 5.3

Example 5.4 *We construct a NURBS* $\mathbf{r}$ *consisting of* 5 *circular arcs. As control points we choose the points*

$$\begin{aligned} &\mathbf{b}_0 = [-10, 0],\ \mathbf{b}_1 = [-6, 1],\ \mathbf{b}_2 = [-1, 1],\ \mathbf{b}_3 = [2, 0], \\ &\mathbf{b}_4 = [7, 1],\ \mathbf{b}_5 = [12, 0.5],\ \mathbf{b}_6 = [14.53582, 0.817]. \end{aligned} \tag{5.13}$$

By hypothesis, $\mathbf{b}_i$ *is the intersection point* $\mathbf{c}_i$ *of the tangents at* K_i *in* $\mathbf{d}_{i-1}$ *and* $\mathbf{d}_i$, $i = 1, \ldots, 5$, *and* $\mathbf{d}_i$ *lies on the straight line from* $\mathbf{b}_i$ *to* $\mathbf{b}_{i+1}$, $i = 1, \ldots, 4$. *Then using (2.2) and (5.13) we determine the points* $\{\mathbf{d}_1, \ldots, \mathbf{d}_4\}$, *the centres* $\{\mathbf{p}_1, \ldots, \mathbf{p}_5\}$ *and the radii* $\{r_1, \ldots, r_5\}$ *of the circular arcs. By a simple calculation we obtain*

$$n_2 = 0.61165,\ n_3 = 0.1638,\ n_4 = 0.20165$$

which implies that

$$1 - n_2(1 - n_4) - n_3 - n_4 > 0.$$

Therefore, by (5.10) there exists a solution of our problem depending on t_3 *and* t_7. *Setting* $t_3 = 1$, $t_7 = 5$ *and using (5.11) and (5.12) we get the weights and the knots of the minimal NURBS* $\mathbf{r}$ *as*

$$\begin{aligned} &w_0 = 1,\ w_1 = 1.03555,\ w_2 = 1.1006,\ w_3 = 1.26625, \\ &w_4 = 1.95243,\ w_5 = 3.49399,\ w_6 = 4.76758 \end{aligned}$$

resp.

$$\begin{aligned} &t_0 = t_1 = t_2 = 0,\ t_3 = 1,\ t_4 = 1.22604,\ t_5 = 1.90381, \\ &t_6 = 3.19043,\ t_7 = t_8 = t_9 = 5 \end{aligned}$$

(all values are rounded). Then defining $\mathbf{r}$ *by (4.1) we obtain the NURBS which describes* $C = \cup_{i=1}^{5} K_i$ *(s. Fig. 5).*

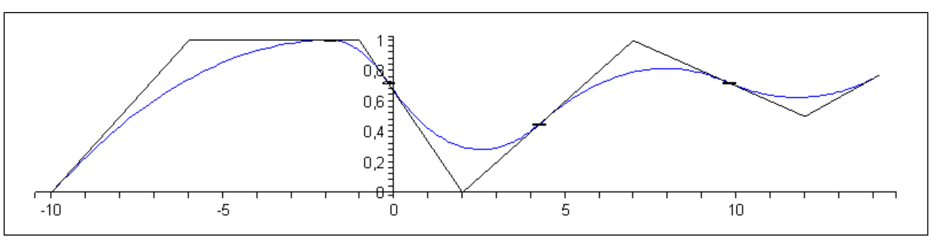

Figure 5: NURBS and control polygon: Example 5.4

Remark 5.5 By (2.1) we have restricted our considerations to circular arcs K_i with $\alpha_i < \pi$. But for K_i with $\pi < \alpha_i < 2\pi$ the construction of a minimal NURBS discussed in the Sections 4 and 5 is also possible. To do it, in the formulas (4.15) - (4.17) we have to set

$$\hat{w}_i = -\cos \beta_i < 0,$$

where again $\beta_i = \sphericalangle(\widetilde{\mathbf{b}}_2, \widetilde{\mathbf{b}}_0, \widetilde{\mathbf{b}}_1) = \sphericalangle(\mathbf{d}_i, \mathbf{d}_{i-1}, \mathbf{b}_i)$. Following the arguments in [2] we can then verify that K_i is represented by the weights $\hat{w}_{i-1} = \hat{w}_{i+1} = 1$ and $\hat{w}_i$ (K_i is the complementary circular arc to the arc with centre angle $2\pi - \alpha_i < \pi$ represented by the weights $\hat{w}_{i-1} = \hat{w}_{i+1} = 1$ and $\cos\beta_i = -\hat{w}_i > 0$). Hence the above discussed construction of a minimal NURBS consisting of circular arcs with $0 < \alpha_i < 2\pi$, $\alpha_i \neq \pi$, is possible. This also holds for the methods studied in Section 3. Analogously as above, if $\alpha_i > \pi$, in (3.4) we have to set

$$\begin{aligned} &w_{2i} = 1, \quad i = 0, \ldots, L, \\ &w_{2i-1} = -\cos\sphericalangle(\mathbf{b}_{2i}, \mathbf{b}_{2i-2}, \mathbf{b}_{2i-1}) < 0, \quad i = 1, \ldots, L. \end{aligned}$$

6 Generalization to Other Types of Conic Sections

It is well known [2] that every conic section, i.e., an ellipse, a parabola or a hyperbola, can be represented as a rational Bézier-curve of degree two with the weights $\hat{w}_{i-1} = \hat{w}_{i+1} = 1$ and $\hat{w}_i \neq 0$ (using the notations in Section 4). Specially, if $0 < \hat{w}_i < 1$, we have an ellipse, if $\hat{w}_i = 1$, we obtain a parabola, and for $\hat{w}_i > 1$, a hyberpola is represented. For the corresponding negative values of $\hat{w}_i$ we obtain the complementary conic sections.

Let us now assume that conic sections $K_1, \ldots, K_L$ are given in $\mathbb{R}^2$ and suppose that there is a tangential transition from K_i to K_{i+1} in $\mathbf{d}_i$ where $\mathbf{d}_{i-1}$ and $\mathbf{d}_i$ denote the endpoints of K_i, $i = 1, \ldots, L$. Moreover, let $\mathbf{c}_i$ be the intersection point of the tangents at K_i in $\mathbf{d}_{i-1}$ and $\mathbf{d}_i$, $i = 1, \ldots, L$. We are interested in constructing a NURBS $\mathbf{r}$ of the type (4.1) which describes the curve

$$\tilde{C} = \bigcup_{i=1}^{L} K_i.$$

Following the arguments in Section 4 we define control points $\{\mathbf{b}_i\}_{i=0}^{L+1}$ by

$$\mathbf{b}_0 := \mathbf{d}_0, \quad \mathbf{b}_i := \mathbf{c}_i, \quad i = 1, \ldots, L, \quad \mathbf{b}_{L+1} := \mathbf{d}_L.$$

We still need the weight $\hat{w}_i$ in the representation of K_i as a Bézier-curve of the type

$$\mathbf{r}(t) = \frac{\hat{w}_{i-1}\widetilde{\mathbf{b}}_0 B_0(\hat{t}) + \hat{w}_i \widetilde{\mathbf{b}}_1 B_1(\hat{t}) + \hat{w}_{i+1}\widetilde{\mathbf{b}}_2 B_2(\hat{t})}{\hat{w}_{i-1} B_0(\hat{t}) + \hat{w}_i B_1(\hat{t}) + \hat{w}_{i+1} B_2(\hat{t})}, \quad t \in I_i, \ \hat{t} \in [0,1],$$

$i = 1, \ldots, L$ (see Section 4, formulas (4.15) and (4.16)).

Then replacing $\cos\beta_i$ by $\hat{w}_i$ in (4.19) and in the following arguments we conclude analogously as in Section 5: Defining the values γ_i, $i = 2, \ldots, L+1$ we can easily verify that Theorem 5.1 characterizes the existence of a minimal NURBS describing $\tilde{C}$ also in the general case.

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