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An Alternative Proof of the Well-Foundedness of the Nested Multiset Ordering

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Abstract

This research note outlines an alternative proof of the well-foundedness of the nested multiset ordering. It is first shown that the set $M^*(S)$ of nested multisets over a given base set S forms a cumulative type structure. Then, by exploiting the notion of sets bounded in rank, it is proved that $(M^*(S), >>^*)$ is well-founded if and only if $(S, >)$ is well-founded, where $>>^*$denotes a nested multiset ordering on $M^*(S)$ and $>$ is an ordering on S.

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1 Introduction

The use of well-founded sets for proving that programs terminate has been suggested by Floyd [3].

A well-founded set (S, >) consists of a set S and an ordering > defined on S such that no infinite descending sequences of elements of S can occur. In other words, a relation being well-founded, every nonempty subset of its domain should have minimal elements. In presence of the axiom of choice, this property is equivalent to there being no infinite descending sequences under the relation. Note that unlike well-ordered sets, well-founded relations may not be unique (DS [1], P.103, for more details).

As to the use of well-founded sets in computer science, essentially the idea is to find a well-founded set and a termination function that maps the elements of the program into that set such that the value of the termination function gets continually reduced throughout the computation. And, as the set considered is well-founded, that value cannot decrease indefinitely, the program must terminate. As a matter of fact, extending a well-founded ordering on a given set into well-founded orderings on data structures built from elements of that set has turned out to be a common tool of the theoretical computer scientist.

Conventionally, the well-founded sets most commonly used for this purpose are the natural numbers under the *greater-than* ordering and n-tuples of natural numbers under the lexicographic ordering.

Dershowitz and Manna ([2], P.189) observe that, in practice, using these conventional orderings often leads to complex termination functions that are difficult to discover; of course, by providing more sophisticated well-founded sets, the corresponding termination functions can be considerably simplified. In the sequel multiset ordering is defined and it is demonstrated how the muliset ordering permits the use of relatively simple and intuitive termination functions in otherwise difficult termination proofs. In the same vein, the concept of the nested multiset ordering is introduced as a generalization of the multiset ordering. We will describe in section 1.1, Dershowitz and Manna's approach to proving the well-foundedness of the nested multiset ordering, and in section 1.2, an alternative approach to that effect.

1.1 Dershowitz and Manna's approach to proving the well-foundedness of the nested multiset ordering.

A multiset is a set that may have multiple occurrences of identical elements. Let $(S, >)$ be a given partially-ordered set and $M(S)$ be the set of all finite multisets over S. In ([2], pp.189-190) the multiset ordering, denoted by $>>$, on $M(S)$ is defined and proved that the multiset ordering $(M(S), >>)$ over $(S, >)$ is well-founded if and only if $(S, >)$ is well-founded. In the same place (P.191), nested multisets over a given base set S are defined as the multisets whose elements may belong to S or be multisets containing both elements of S and multisets of S, and so on. A nested multiset ordering, denoted by $>>^*$, on $M^*(S)$, the set of all nested multisets over S, is defined recursively. That is, $M^0(S) = S$, $M^{i+1}(S)$ contains the multisets whose elements are taken from $M^0(S), M^1(S), \ldots, M^i(S)$, with at least one element taken from $M^i(S)$; where $M^i(S)$ denotes the set of all nested multisets of depth i. Thus, $M^*(S)$ is the infinite union of the disjoint sets $M^0(S), M^1(S), M^2(S), \ldots$. The following property holds: For $M, N \in M^*(S)$, if the depth of M is greater than the depth of N, then $M >>^* N$. In other words, the multisets of $M^i(S)$ are all greater than the multisets of $M^j(S)$, under $>>*$, for any $j < i$.

The following theorem is proved: $(M^*(S), >>^*)$ over $(S, >)$ is well-founded if and only if $(S, >)$ is well-founded. It is further shown that $(M^*(S), >>)$ is well-founded by way of showing that each $M^i(S)$ is itself well-founded under $>>^*$ and the latter is proved by induction on i.

1.2 An alternative approach to proving well-foundedness of the nested multiset ordering.

An alternative proof that $(M^*(S), >>^*)$ is well-founded if and only if $(S, >)$ is well-founded is outlined below.

Cumulative type structure and the rank of a set:

The set of level 0 are individuals and the sets of level n are collections, all of whose members are of levels $< n$. Accordingly, if a set exists at some level, that same set is available at every higher level (to be a member of some new set which exists at that higher level), and it could be viewed as being formed anew at every subsequent level. But for each set there will always be a *first level* at which it exists (namely the first level after all its members exist), and this first level is called the rank of the set ([1], P.7).

The vonNeumann universe:

The vonNeumann universe, V, is constructed inductively, starting from ϕ, the empty set, by successively applying the power-set operation P viz.,

$V_0 = \phi, V1 = P(\phi) = \{\phi\}, V_2 = P(V_1) = \{\phi, \{\phi\}\}, \ldots, V_{n+1} = P(V_n)$.

In order to go beyond finite:

where $V_{\omega_0} = \bigcup_{n=0}^{\infty} V_n, \quad V_{\omega_{0+1}} = P(\ V_{\omega_0}), \cdots$;where $\omega_0, \omega_{0+1}, \cdots$ are infinite ordinals. The formal expression to describe that "the sets bounded in rank" or equivalently, "restricting sets to be in the vonNeumann universe V", is the following:

$(\forall x)(\exists$ an ordinal $\alpha)(x \in V\alpha)$, see ([4] for details).

Clearly, based on the aforesaid definitions, the sequence $M^i(S)$, the set of all nested multisets of depth i on a given finite base set S, forms a cumulative type structure and in turn, also belongs to the vonNeumann universe. Accordingly, each $M^i(S), i = 0, 1, 2, \cdots$, is bounded in rank and any infinite descending chain must terminate; for example, it terminates to ϕ under the elementhood relation. Hence, in presence of the axion of choice, it follows that$M^*(S)$ is well-founded.

Moreover, $M^*(S) = V_{\omega_0} = \bigcup_{i=0}^{\infty} M^i(S)$, the infinite union of $M^0(S), M^1(S), \cdots$. Here, ω_0 is the smallest transfinite ordinal number. All other details of the proof can be mimicked from [2].

In conclusion, we can say that in order to construct an appropriate termination function disallowing the occurrence of infinite descending chain, the strategy to be constantly observed is that no operations involved should take us outside the vonNeumann universe. In other words, at each step of construction, all the outcomes need to be necessarily bounded in rank.

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