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Reproductive Number

S. Seddighi Chaharborj[1,*], M. R. Abu Bakar[1], A. H. Malik[1] and V. Alli[2]

[1]Department of Mathematics, Faculty of Science, UPM
University Putra Malaysia, 43400 UPM

[2]Mathematics Department, Ministry of Science and Technology, Tehran, Iran

[*] sseddighi@aeoi.org.ir (S. Seddighi)

Abstract

In this article we want to find the reproductive number, R_0, for the three models with a simple method. With this simple method we can find the reproductive number for all of models.

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1 Introduction

One of the fundamental questions of mathematical epidemiology is to find threshold conditions that determine whether an infectious disease will spread in a susceptible population when the The disease is introduced into the population. The threshold conditions are characterized by the so-called reproductive number, the reproduction number, the reproductive ratio, basic reproductive value, basic reproductive rate, or contact number, commonly denoted by R0 in mathematical epidemiology [1]. The concept of R_0, introduced by Ross in 1909 [2], is defined in epidemiological modeling such that if $R_0 < 1$, the modeled disease dies out, and if $R_0 > 1$, the disease spreads in the population[1].

In this paper we try to find reproductive number for many of models with a simple method.

2 DI Model

The dynamics of the transmission are governed by the following system[1]:

$$\begin{cases} \frac{dS}{dt} = \mu(S^0 - S) - \lambda S, \\ \frac{dI_i}{dt} = p_i \lambda S - (\mu + \nu_i) I_i, \quad i = 1, ..., n, \\ \frac{dA}{dt} = \sum_{i=1}^{n} \nu_i I_i - \delta A, \end{cases} \tag{1}$$

$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t)$ and $\lambda_i(t) = r\beta_i(I_i(t)/N(t))$,
where $N(t) = S(t) + \sum_{i=1}^{n} I_i(t)$.

The reproductive number for this model is,

$$R_0 = r \sum_{i=1}^{n} \frac{p_i \beta_i}{\mu + \nu_i}.$$

Now, we want to find the reproductive number for this model with a simple method. In this method for find R_0, we survey to have increase and decrease of infectives, therefore we have,

$$i)\ I_i^{\prime} > 0 \implies p_i \lambda S - (\mu + \nu_i) I_i > 0 \implies (\mu + \nu_i) I_i < p_i (r \sum_{i=1}^{n} \beta_i \frac{I_i}{N}) S$$

Now, In the disease-free equilibrium ($S_i = S_i^0, I_i = 0, i = 1, ..., n$), we have:

$$\begin{aligned} (\mu + \nu_i) I_i \quad &< \quad (r p_i \sum_{i=1}^{n} \beta_i) I_i \quad \implies (\mu + \nu_i) < (r p_i \sum_{i=1}^{n} \beta_i) \\ &\implies \quad r \sum_{i=1}^{n} \frac{p_i \beta_i}{\mu + \nu_i} > 1 \end{aligned}$$

If we take $R_0 = r \sum_{i=1}^{n} (p_i \beta_i)/(\mu + \nu_i)$, then for $I_i^{\prime} > 0$ we have, $R_0 > 1$. Also for $I_i^{\prime} < 0$ we have, $R_0 < 1$.

3 SP Model

The equations for the **SP** model are[1],

$$\begin{cases} \frac{dS}{dt} = \mu(S^0 - S) - \lambda S, \\ \frac{dI_1}{dt} = \lambda S - (\gamma_1 + \mu)I_1, \\ \frac{dI_i}{dt} = \gamma_{i-1}I_{i-1} - (\gamma_i + \mu), & 2 \leq i \leq n, \\ \frac{dA}{dt} = \gamma_n I_n - \delta A, \end{cases} \tag{2}$$

$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t)$ and $\lambda_i(t) = r\beta_i(I_i(t)/N(t))$.

The reproductive number for this model is,

$$R_0 = r\sum_{i=1}^{n} \frac{\beta_i q_i}{\mu + \gamma_i},$$

where $q_i := \prod_{j=1}^{i-1} \gamma_j/(\mu + \gamma_j)$.

Now, we want to find the reproductive number for this model with the simple method. Therefore, we have:

i) $\acute{I_i} > 0 \Longrightarrow \acute{I_1} > 0 \Longrightarrow \lambda S - (\gamma_1 + \mu)I_1 > 0,$

$$\Longrightarrow I_1 < \frac{\lambda S}{\gamma_1+\mu},$$
$$I_2 < \frac{\gamma_1}{\gamma_2+\mu} I_1 < (\frac{\gamma_1}{\gamma_2+\mu})(\frac{\lambda S}{\gamma_1+\mu}),$$
$$I_3 < (\frac{\gamma_2}{\gamma_2+\mu})(\frac{\gamma_1}{\gamma_2+\mu})(\frac{\lambda S}{\gamma_1+\mu}),$$
$$\vdots$$
$$I_i < (\frac{\gamma_{i-1}}{\gamma_i+\mu}) \cdots (\frac{\gamma_2}{\gamma_2+\mu})(\frac{\gamma_1}{\gamma_2+\mu})(\frac{\lambda S}{\gamma_1+\mu})$$
$$\Longrightarrow I_i < (\frac{\gamma_1}{\gamma_1+\mu})(\frac{\gamma_2}{\gamma_2+\mu}) \cdots (\frac{\gamma_{i-1}}{\gamma_{i-1}+\mu})(\frac{\lambda S}{\gamma_i+\mu}),$$

Now, In the disease-free equilibrium $(S_i = S_i^0, I_i = 0, i = 1, ..., n)$, we have:

$$I_i < r\sum_{i=1}^{n} \frac{\beta_i q_i}{\gamma_i + \mu} I_i \quad \Longrightarrow 1 < r\sum_{i=1}^{n} \frac{\beta_i q_i}{\gamma_i + \mu},$$

with $q_i := \prod_{j=1}^{i-1} \gamma_j/(\mu + \gamma_j)$.

If we take $R_0 = r\sum_{i=1}^{n}(\beta_i q_i)/(\gamma_i + \mu)$, then for $\acute{I_i} > 0$ we have, $R_0 > 1$. Also for $\acute{I_i} < 0$ we have, $R_0 < 1$.

4 The combined DS and DI Models

The dynamics of the transmission are governed by the following system[1]:

$$\begin{cases} \frac{dS_i}{dt} = \mu(S_i^0 - S_i) - \lambda_i S_i, \quad i = 1, ..., n, \\ \frac{dI_j}{dt} = \sum\limits_{k=1}^{n} p_{kj}\lambda_k S_k - (\mu + \nu_j)I_j, \quad j = 1, ..., m, \\ \frac{dA}{dt} = \sum\limits_{k=1}^{m} \nu_k I_k - \delta A, \end{cases} \tag{3}$$

with

$$\lambda_i = r\alpha_i \sum_{j=1}^{m} \beta_j \frac{r_j I_j}{r\sum\limits_{l=1}^{n} S_l + \sum\limits_{k=1}^{m} r_k I_k}.$$

The reproductive number for this model is,

$$R_0 = \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{p_{kj}\alpha_k S_k^0 r_j \beta_j}{(\mu + \nu_j)\sum\limits_{l=1}^{n} S_l^0}.$$

Now, we want to find the reproductive number for this model with the simple method. Therefore, we have:

$$i)\ \acute{I_j} > 0 \Longrightarrow I_j < \frac{\sum\limits_{k=1}^{n} p_{kj}\lambda_k S_k}{\mu+\nu_j} = \frac{\sum\limits_{k=1}^{n} p_{kj}(r\alpha_k \sum\limits_{j=1}^{m} \beta_j \frac{r_j I_j}{r\sum\limits_{l=1}^{n} S_l + \sum\limits_{k=1}^{m} r_k I_k})S_k}{\mu+\nu_j},$$

$$\Longrightarrow I_j < r\sum_{j=1}^{m}\sum_{k=1}^{n} \frac{p_{kj}\alpha_k r_j \beta_j S_k}{(\mu+\nu_j)(r\sum\limits_{l=1}^{n} S_l + \sum\limits_{k=1}^{m} r_k I_k)} I_j,$$

$$\Longrightarrow 1 < r\sum_{j=1}^{m}\sum_{k=1}^{n} \frac{p_{kj}\alpha_k r_j \beta_j S_k}{(\mu+\nu_j)(r\sum\limits_{l=1}^{n} S_l + \sum\limits_{k=1}^{m} r_k I_k)}.$$

Now, In the disease-free equilibrium ($S_i = S_i^0, I_j = 0, i = 1, ..., n, j = 1, ..., m$), we have:

$$1 < \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{p_{kj}\alpha_k r_j \beta_j S_k^0}{(\mu + \nu_j)(\sum\limits_{l=1}^{n} S_l^0)}$$

If we take $R_0 = \sum\limits_{j=1}^{m}\sum\limits_{k=1}^{n}(p_{kj}\alpha_k r_j \beta_j S_k^0)/((\mu + \nu_j)(r\sum\limits_{l=1}^{n} S_l^0))$, then for $\acute{I_j} > 0$ we have, $R_0 > 1$. Also for $\acute{I_j} < 0$ we have, $R_0 < 1$.

References

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