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A Note on Cylindric Algebras

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Abstract

If two atomic cylindric set algebras of finite dimension > 1 are isomorphic, then base-minimality does not imply that they are lower-base isomorphic. This contrasts the case of boolean algebras.

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Let $\mathfrak{A} \subseteq \wp(U)$ and $\mathfrak{B} \subseteq \wp(V)$ be two boolean algebras (BA's). It is natural to ask the following question: For what kinds of BA's does $\mathfrak{A} \cong \mathfrak{B}$ imply the existence of $f : U \to V$ such that f induces an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Such isomorphisms are called base - isomorphisms and if such an isomorphism exists then $\mathfrak{A}$ and $\mathfrak{B}$ are said to be base isomorphic.

$\mathfrak{A}$ is called *base minimal* if for no proper $Z \subset U$ is the function

$$rl_Z = \langle X \cap Z : X \in A \rangle$$

an isomorphism on $\mathfrak{A}$. (This means that the base U of $\mathfrak{A}$ is minimal in the sense that if we omit any element from it then this would change the isomorphism type of $\mathfrak{A}$.) It is known that

(*) If the BA's $\mathfrak{A}$ and $\mathfrak{B}$ are both atomic and base - minimal then $\mathfrak{A} \cong \mathfrak{B}$ implies that they are base - isomorphic.

We show that (*) does not generalize to finite dimensional cylindric set algebras. To formulate our Theorem, we need some definitions. We follow the notation of [4]. In particular $\mathbf{CA}_n(\mathbf{Cs}_n)$ denotes the class of cylindric (set) algebras of dimension n. Let $\mathfrak{A} \in \mathbf{Cs}_n$ with base U (i.e with greatest element ${}^nU$.) let $f : U \to W$ be a bijection. For any $X \in \mathfrak{A}$, let

$$\bar{f}X = \{y \in {}^nW : f^{-1} \circ y \in X\}.$$

Then $\bar{f}$ is the base-isomorphism induced by f. Assume that $\mathfrak{B} \in \mathbf{Cs}_n$ with base W. $\mathfrak{A}$ and $\mathfrak{B}$ are base-isomorphic if there is a bijection mapping U onto W such that $\bar{f} \upharpoonright \mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. Let $V \subseteq {}^nU$. Let

$$rl_V = \langle X \cap V : X \in A \rangle.$$

If rl_V is an isomorphism and $V = {}^nU'$ for some $U' \subseteq U$, then rl_V is called a strong ext-isomorphism. $\mathfrak{A}$ is base-minimal if it is not strongly ext-isomorphic to a $\mathbf{Cs}_n$ except itself. A function g is a lower-base isomorphism if $g = k^{-1} \circ h \circ t$ for some strong ext-isomorphisms k and t and some base isomorphism h. Two algebras are lower base isomorphic if there exists a lower base isomorphism between them. (Using the terminology of [5] a lower base isomorphism is a composition of a strong ext-isomorphism with a subbase isomorphism. So two $\mathbf{Cs}$'s are lower base isomorphic if they are both ext-base isomorphic to a third one.)

We shall prove;

Theorem . *Let $n > 1$. Then there exist infinitely many pairwise isomorphic finite cylindric set algebras of dimension n, that are base minimal but any two of which are not lower base-isomorphic.*

There is an infinite version of (*) for (locally finite) cylindric algebras that has been thoroughly investigated [6], [8], [9] and [3]. There are cases where isomorphism and base minimality imply lower base isomorphism but it is not always the case. The cardinality of the algebras involved play a key role. The logical counterpart of this problem has to do with the existence and uniqueness of atomic or prime models for atomic theories. This problem - under which conditions isomorphism implies lower base isomorphism - was investigated thoroughly by Biro [2], but to the best of our knowlegde he did not address the case dealt with here.

1 Proof

Throughout n is a natural number > 1.

Notation .

(i) The full cylndric set algebra with base U and dimension n is denoted by $\mathfrak{A}(n, U)$. Full here means that the universe of $\mathfrak{A}(n, U)$ is $\wp({}^nU)$, the power set of ${}^nU$.

(ii) For a given $\mathbf{CA}_n$, we let d_n, or even sometimes simply d, stand for the principal diagonal element that is

$$\mathsf{d}_n = \mathsf{d} = \prod\{\mathsf{d}_{kl} : k, l < n\}.$$

We let $\bar{\mathsf{d}}_n$ *stand for the principal co-diagonal that is*

$$\bar{\mathsf{d}}_n = \prod\{-\mathsf{d}_{kl} : k, l < n, k \neq l\}.$$

(iii) Let σ *be a permutation of the base* U *of* $\mathfrak{A}(n, U)$*. Then* σ *induces an automorphism on* $\mathfrak{A}(n, U)$*; which we denote by* $\bar{\sigma}$*, or sometimes also by* σ*, when no confusion is likely to ensue. More specifically for* $X \subseteq {}^nU$*, and* σ *a permutation of* U*, we let*

$$\sigma(X) = \{\sigma \circ y : y \in X\}.$$

(iv) The symmetric group on a set U *is denoted by* S_U*. In particular, the universe of* S_U *is the set of all bijections from* U *to* U*.*

(v) For an algebra $\mathfrak{A}$ *and* $X \subseteq A$*,* $\mathfrak{Sg}^{\mathfrak{A}}X$ *or simply* $\mathfrak{Sg}X$ *when* $\mathfrak{A}$ *is clear from context, denotes the subalgebra of* $\mathfrak{A}$ *generated by* X*.* Id_X *denotes the identity map with domain* X*; the subscript* X *will be dropped when the domain is clear.* $|X|$ *denotes the cardinality of* X*.* $Rg(f)$ *for a given function* f *denotes the range of* f*.* $f \upharpoonright X$ *denotes the restriction of* f *to* X*. The composition of the functions* f *and* g *is defined so that the righthand most function acts first. That is* $f \circ g(x) = f(g(x))$ *whenever* $g(x) \in Rg(f)$*. For* $s \in {}^n\mu$*,* $i < n$ *and* $a \in \mu$*, we let* $s(i|a)$ *denote the function for which* $s(i|a)(i) = a$ *and* $s(i|a)(j) = s(j)$ *whenever* $j \neq i$*.*

From now on, unless otherwise specified, it is assumed that n is a natural number > 1, μ is a cardinal such that $n + 2 \leq \mu$, $H \subseteq \mu$ and H is finite. For $x \in H$, we let

$$a_x = n \times \{x\} = \{\langle x : i < n\rangle\}$$

be the atom of $\mathfrak{A}(n, \mu)$ corresponding to the function with constant value x. Let $\mathfrak{A}(n, \mu, H)$ denote the subalgebra of $\mathfrak{A}(n, \mu)$ generated by the set $\{a_x : x \in H\}$. In particular, this subalgebra contains the elements $h(n, \mu, H) = {}^nH$ and $\bar{h}(n, \mu, H) = h(n, \mu, H) \cdot \bar{\mathsf{d}}_n$. When n, μ and H are clear from context, we write $h = h(n, \mu, H)$ and $\bar{h} = \bar{h}(n, \mu, H)$ for short.

To formulate the next two lemmas we need the notion of two elements of a cylindric algebra always belonging to the same cylinders. Namely, X, Y in a $\mathbf{CA}_n$ are called *cylindrically equivalent* iff $\mathsf{c}_iX = \mathsf{c}_iY$ for all $i < n$. We also need the following notation. For $X \in A$ and $\mathfrak{A} \in \mathbf{Cs}_n$, we let $S_{[0,1]}X = \{s \in {}^nU : s \circ [0, 1] \in X\}$. Here $[0, 1]$ is the transposition on n that interchanges 0 and 1. The next Lemma is the key technical result in this paper.

Lemma 1 . *For* $4 \leq n+2 \leq |H| < \omega$ *with* $H \subseteq \mu$*, there exists* $X \in A(n, \mu, H)$ *for which* $X \subseteq \bar{h}$ *and* (1)*,* (2)*,* (3) *and* (4) *below hold:*

(1) $\{\sigma \in S_H : \bar{\sigma}(X) = X\} = \{Id_H\}$.

(2) Both X *and* $\bar{h} \sim X$ *are cylindrically equivalent to* $\bar{h}$.

(3) $|X| \neq |\bar{h} \sim X|$.

(4) Further more, if $\mu = H$, *then* $\bar{h} = \bar{d}$, $\mathfrak{A}(n, \mu, H) = \mathfrak{A}(n, \mu)$, *the full set algebra with base* μ, *and the following condition holds:*

$$S_{[0,1]}X \notin \{X, \bar{d} \sim X\}.$$

Proof. We start by proving (1), (2) and (3). This will be done by induction on n. Then we prove (4). Since $\mathfrak{A}(n, \mu, H) \cong \mathfrak{A}(n, \mu, H')$ whenever $|H| = |H'|$, we can assume without loss of generality that H is an initial segment of μ. i.e.

$$H = m = \{0, \dots, m-1\}, \text{ where } |H| = m.$$

Now the base step of the induction is easy.

For $n = 2$ and $4 \leq m < \omega$, we let

$$X = \{s \in {}^2m : s_1 = s_0 + 1(mod(m))\} \cup \{(0, 2)\}.$$

Then it is not hard to check that $X \subseteq \bar{h}$ and that (1), (2), and (3) hold.

For the induction step we assume that $X \subseteq \bar{h}(n, \mu, m)$ has been defined satisfying (1),(2) and (3), and we define $\bar{X} \subseteq \bar{h}(n+1, \mu+1, m+1) = \bar{H}$(for short) which also satisfies (1), (2) and (3). First we define

$$N = \{s \in \bar{H} : m \notin Rg(s)\};$$

and for $i \leq n$, we let

$$Z_i = \{s \in \bar{H} : s_i = m\}.$$

Then the set $\{N\} \cup \{Z_i : i \leq n\}$ forms a partition of $\bar{H}$ (which will be used to separate cases). We define for each $i \in n$

$$A_i = \{s \in Z_i : s(i|s_n) \upharpoonright n \in X\}.$$

and we let

$$\bar{X} = \{s \in N : s \upharpoonright n \in X\} \cup \{s \in Z_n : s \upharpoonright n \notin X\} \cup \cup_{i \in n} A_i.$$

We show that $\bar{X}$ is as desired. Clearly $\bar{X} \subseteq \bar{H}$. We now consider (2), then (3), and then (1).

Proof of (2)

Suppose $i \in n+1$. Clearly $\mathsf{c}_i\bar{X} \subseteq \mathsf{c}_i\bar{H}$. We show the reverse inclusion, namely $\bar{H} \subseteq \mathsf{c}_i\bar{X}$. Towards, this end, assume that $s \in \bar{H}$. We must show that $s \in \mathsf{c}_i\bar{X}$. We distinguish between two cases

Case 1. $s \in N \cup Z_n$.

Subcase 1.1. $i \in n$

If $s \in N$, then $m \notin Rg(s)$, and so $s \upharpoonright n \in h$. Also, if $s \in Z_n$, then $s(n) = m$, and since s is one to one, it follows that $s(j) \neq m$ for all $j < n$, hence we also have $s \upharpoonright n \in h$. Since $s \upharpoonright n \in \bar{h}(n, \mu, m)$ and by induction we have $\bar{h}(n, \mu, m) \subseteq \mathsf{c}_iX$, it follows that there is an $a \in m$ such that

$$[s \upharpoonright n](i|a) = s(i|a) \upharpoonright n \in X.$$

One of two things. Either $a \neq s_n$ or $a = s_n$. In the former case we get $s(i|a) \in \bar{X}$. In the latter we have $s(i|m) \in \bar{X}$ since $s(i|m) \in Z_i$ and

$$s(i|m)(i|s_n) \upharpoonright n = s(i|a) \upharpoonright n \in X.$$

We have proved that $s \in \mathsf{c}_i\bar{X}$. Therefore $\bar{X}$ satisfies (2).

Subcase 1.2. $i = n$.

If $s \upharpoonright n \notin X$, then by definition $s(n|m) \in \bar{X}$. Else $s \upharpoonright n \in X$. Since $m > n+1$, there exists $a \in m \sim Rg(s \upharpoonright n)$. Thus $s(n|a) \in N$, which in turn implies that $s(n|a)$ is in $\bar{X}$. In either case we get that $s \in \mathsf{c}_n\bar{X} = \mathsf{c}_i\bar{X}$.

Case 2. $s \in Z_k$, $k \in n$ and $i \in \{k, n\}$.

Subcase 2.1. $i = k$

Since $\bar{h} \subseteq \mathsf{c}_kX$, there exists $a \in m$ such that $s(k|a) \upharpoonright n \in X$. Thus, $s(n|a) \in \bar{X}$ so $s \in \mathsf{c}_n\bar{X}$. If $a \neq s_n$, then $s(k|a) \in \bar{X}$; otherwise $s \in \bar{X}$. In either case, $s \in \mathsf{c}_k\bar{X}$.

Subcase 2.2. $i = n$

Since $s(k|s_n) \upharpoonright n \in \bar{h} \subseteq \mathsf{c}_iX$ there exists $a \in m$ such that $s(k|s_n)(i|a) \upharpoonright \in X$. Thus $s(i|a) \in \bar{X}$, and so $s \in \mathsf{c}_i\bar{X}$ as desired.

By Cases 1 and 2 above it follows that $\bar{H} \subseteq \mathsf{c}_i\bar{X}$ hence $\mathsf{c}_i\bar{H} = \mathsf{c}_i\bar{X}$. The proof that $\mathsf{c}_i\bar{H} = \mathsf{c}_i(\bar{H} \sim \bar{X})$ is completely analogous and is therefore omitted. By this we have proved that $\bar{X}$ satisfies (2).

Proof of (3)

We now prove that $\bar{X}$ satisfies (3). For the sake of brevity, we write $Y = \bar{h} \sim X$, $\bar{Y} = \bar{H} \sim \bar{X}$, and we set for each $a \leq m$

$$X_a = \{s \in \bar{X} : s_n = a\},$$

and for $i \in n$

$$B_i = \{s(i|m)(n|a) : s \upharpoonright n \in X \text{ and } s_i = a\}.$$

Note that

$$|X_m| = |\{s \in \bar{H} : s \upharpoonright n \in Y \text{ and } s_n = m\}| = |Y|.$$

For $a \in m$, we have

$$X_a = \{s \in \bar{H} : s \upharpoonright n \in X \text{ and } a \notin Rg(s \upharpoonright n)\} \cup \cup\{B_i : i \in n\}$$

showing that $|X_a| = |X|$. Thus, $|\bar{X}| = |Y| + m|X|$. An analogous argument shows that $|\bar{Y}| = |X| + m|Y|$, so $|\bar{X}| - |\bar{Y}| = (m-1)(|X| - |Y|)$ from which it follows that $|X| \neq |Y|$ which in turn implies that $|\bar{X}| \neq |\bar{Y}|$.

Proof of (1)

We now prove that $\bar{X}$ satisfies (1). Suppose $\sigma \in S_{m+1}$ and that $\sigma \neq Id$. We distinguish between two case:

Case 1. $\sigma(m) = m$.

In this case $\sigma \upharpoonright m = \tau \in S_m$, $\tau \neq Id$. Since by induction (1) holds for X, $\bar{\tau}X \neq X$ and thus $\bar{\tau}Y \neq Y$. Choose $p \in Y$ for which $\tau(p) \notin Y$ and set $f = p \cup \{(n,m)\}$. Then $f \in \bar{X}$ while $\sigma(f) = \tau(p) \cup \{(n,m)\} \notin \bar{X}$.

Case 2. $\sigma(a) = m$ for some $a \in m$.

If $\bar{\sigma}\bar{X} = \bar{X}$ then $\bar{\sigma}X_a = X_m$. But $|\sigma X_a| = |X| \neq |Y| = |X_m|$ by (3), and so $\bar{\sigma}\bar{X} = \bar{X}$. This complete the proof of the induction step, hence $\bar{X}$ satisfies $(1), (2)$ and (3).

Now we prove (4). We distinguish between the case $n = 2$ and the case $n > 2$. Let $n = 2$. Let X be as defined above in the base step of the induction. Then we have $(1,2) \in X$ but $(2,1) \notin X$. This shows that $S_{[0,1]}X \neq X$. Also $(1,3) \notin X$ and $(3,1) \notin X$, hence $S_{[0,1]}X \neq \bar{d} \sim X$. Now let $2 < n$. Let $p = \{(0,1),(1,2),(3,0)\} \cup \{(i, m+i-2) : 3 \leq i < n\}$. Then $p \in {}^n m$. For $2 \leq \beta \leq n$, we denote X and N, constructed above in the induction step for dimension β by X_β and N_β, respectively. It is not hard to see that by the construction of X_β, we get

$$(\forall 2 \leq \beta < n)(p \upharpoonright \beta + 1 \in X_{\beta+1} \text{ iff } p \upharpoonright \beta \in X_\beta).$$

In particular, $p \in X_n$ iff $p \upharpoonright 2 \in X_2$. It follows thus that $p \in X_n$ but $S_{[0,1]}p \notin X_n$. Thus $S_{[0,1]}X_n \neq X_n$. Now let $q = \{(0,1),(1,3),(3,0)\} \cup \{(i, m+i-2) : 3 \leq i < n\}$. Then it is easy to see that $q \notin X_n$ and $S_{[0,1]}q \notin X_n$. Thus $S_{[0,1]}X_n \neq \bar{\mathsf{d}} \sim X_n$. By this the proof of Lemma 1 is complete. ■

For a boolean algebra with extra operations $\mathfrak{A}$, $At\mathfrak{A}$ denotes the set of atoms of $\mathfrak{A}$. (Recall that an atom is a minimal non-zero element). $Bl\mathfrak{A}$ denotes its boolean reduct (which is a boolean algebra).

Lemma 2 . *Suppose $\mathfrak{A}$ is a finite subalgebra of $\mathfrak{C} \in \mathbf{CA}_n$. Let W be an atom of $Bl\mathcal{A}$ and let Y be a partition of W into finitely many elements, each of which is cylindrically equivalent to W. Let $\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{C}}(A \cup Y)$. Then (i) and (ii) below hold:*

(i) $Y \subseteq At(\mathfrak{B})$

(ii) $\mathfrak{B} = \mathfrak{Sg}^{Bl\mathfrak{C}}(A \cup Y)$, *i.e* $\mathfrak{B}$ *coincides with the boolean subalgebra of* $\mathfrak{C}$ *generated by* $A \cup Y$.

Proof. For the sake of brevity, let $\mathfrak{D} = \mathfrak{Sg}^{BlC}(A \cup Y)$. We first show that $D = B$. Clearly $D \subseteq B$, since B is closed under the boolean operations. Since D, by definition, is closed under the boolean operations and contains all the diagonal elements, to show that $B \subseteq D$ it remains to show that D is closed under cylindrifications. Towards this end, let

$$Z = (At(\mathcal{A}) \sim \{W\}) \cup Y.$$

Then $\mathfrak{D} = \mathfrak{Sg}^{Bl\mathfrak{C}} Z$ because $W = \sum Y$ and because $\mathfrak{A}$ is generated as a boolean algebra by its atoms. Therefore

(iii) $Y \subseteq At(\mathcal{D})$

and

(iv) Every element of D is a sum of a subset of Z.

Now for each $z \in Z$, $\mathsf{c}_i z \in A$ because $A \subseteq C$ and for each $y \in Y$, we have $\mathsf{c}_i y = \mathsf{c}_i W \in A$. Since c_i is additive (iv) implies $\mathsf{c}_i b \in A$ for all $b \in D$. It follows that $D = B$ which proves (ii). Now (i) readily follows from (iii). ■

We now prove our Theorem: Fix $n > 1$. Let $l > 1$. Let us take the special case when $\mu = H = n + l$ and let $X \in \mathfrak{A}(n, n+l)$ be as specified in Lemma 1. We sometimes denote X by X_n^l to emphasize the roles of n and l. Note that $X_n^l \in \mathfrak{A}(n, n+l)$. Now Let

$$\mathfrak{B}_n^l = \mathfrak{Sg}^{\mathfrak{A}(n,n+l)}\{X_n^l\}.$$

We shall show that for fixed $n > 1$ the system $\langle \mathfrak{B}_n^l : l > 1 \rangle$ consists of pairwise isomorphic algebras that are base minimal but not lower base- isomorphic. We first show that $\mathfrak{B}_n^l$ is base-minimal. We fix l and proceed by induction on n. The case $n = 2$ is easy. Now suppose that the statement is true for n. We shall show that it is true for $n + 1$. Let $W \subseteq n + 1 + l$ be such that

$$\langle Y \cap {}^{n+1}W : Y \in \mathfrak{B}_{n+1}^l \rangle$$

is an isomorphism. For brevity, write V for ${}^{n+1}W$, X for X_n^l, X^+ for X_{n+1}^l and $\mathfrak{R}$ for $rl_V \mathfrak{B}_{n+1}^l$. We claim that

$$n + l \in W.$$

If not, then

$$\mathfrak{B}_{n+1}^l \models \prod\{-\mathsf{d}_{jk} : j < k < n + 1\} \neq 0.$$

Hence

$$\mathfrak{R} \models \prod\{-\mathsf{d}_{jk} : j < k < n + 1\} \neq 0.$$

Now let

$$q \in \bigcap\{V \sim \mathsf{d}^V_{jk} : j < k < n+1\}.$$

Then

$$\begin{aligned} q \in \mathsf{c}^{\mathfrak{R}}_n \bar{\mathsf{d}}_{n+1} &= \mathsf{c}^{\mathfrak{B}^l_{n+1}}_n \bar{d}_{n+1} \cap V \\ &= \mathsf{c}^{\mathfrak{B}^l_{n+1}}_n (\bar{\mathsf{d}} \sim X^+) \cap V \\ &= \mathsf{c}^{\mathfrak{R}}_n (\bar{\mathsf{d}} \sim (X^+ \cap V)). \end{aligned}$$

Let $u \in W$ such that

$$q(n|u) \in \bar{\mathsf{d}}_{n+1} \sim X^+.$$

Let

$$N = \{s \in \bar{\mathsf{d}}_{n+1} : n+1 \notin Rg(s)\}$$

then

$$q(n|u) \in N.$$

So

$$q \restriction n \in X$$

thus

$$q(n|n+l) \in X^+.$$

Therefore $q \in \mathsf{c}^{\mathfrak{B}^l_{n+1}}_n X^+ \cap V = \mathsf{c}^{\mathfrak{R}}_n X^+$, so there is a $t \in W$ such that $q(n|t) \in X^+$. But the only choice of t is $n+l$ hence $n+l \in W$. Now let

$$\mathfrak{B} = \mathfrak{B}^l_n$$

$$W' = W \sim \{n+l\}$$

$$V' = {}^nW'.$$

Then we claim that $rl^{\mathfrak{B}}_{V'}$ s an isomorphism. It is enough to show that $Y \cap V' \neq \emptyset$ whenever $Y \in At\mathfrak{B}$. If $Y \in At\mathfrak{B}$ then either Y is an atom of the minimal subalgebra of $\mathfrak{B}$ or $Y = X$ or $Y = \bar{d} \sim X$. (This follows from the fact that the latter two elements are cylindrically equivalent.) We have $|W| \geq n+1$ so $|W'| \geq n$. Hence if Y is an atom of the minimal subalgebra of $\mathfrak{B}$, then $Y \cap V' \neq \emptyset$. Now assume that $Y = X$. Take $q \in V \cap \bar{d}_{n+1}$ such that $q(n) = n+l$. Then

$$\begin{aligned} q &\in \mathsf{c}^{\mathfrak{B}^l_{n+l}}_n (\bar{d}_{n+1} \sim X^+) \cap \bar{d}_{n+1} \cap V. \\ &= \mathsf{c}^{\mathfrak{R}}_n (\bar{d}_{n+1} \sim X^+) \cap \bar{d}_{n+1}. \end{aligned}$$

It follows that there exist $q'_0, \dots q'_{n-1} \in W'$ such that $q' = (q'_0, \dots q'_{n-1}, n+l) \in (\bar{d}_{n+1} \sim X^+) \cap V$, thus $q' \restriction n \in X \cap V'$. The last case can be treated similarly. Now by the induction hypothesis we get $W' = n+l$. Thus $W = n+l+1$. We now show that for $i \neq j$, $\mathfrak{B}^i_n \cong \mathfrak{B}^j_n$. Let f be the permutation of the atoms that sends X_n^i to X_n^j, $\bar{d} - X_n^i$ to $\bar{d} - X_n^j$, and the diagonal d_{kl} to the diagonal d_{kl}, for $k, l < n$. Then f extends to an isomorphism of the boolean parts of the algebras $\mathfrak{B}_n^i$ and $\mathfrak{B}_n^j$, and preserves the diagonal elements. By Lemma 2 it preserves cylindrifications as well. Also for $i \neq j$, $\mathfrak{B}_n^i$ and $\mathfrak{B}_n^j$ cannot be lower base isomorphic since $n + i \neq n + j$ and they are base minimal.

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