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A Theorem on Semilattice-Ordered Semigroup[1]

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Abstract

Semilattice-ordered semigroup is important algebraic structure. It is equipped with the natural defined positive quasi-antiorder relation.

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1 Introduction and preliminaries

This investigation is in Bishop's constructive algebra in sense of papers [9]-[12] and books [7] and [13] (Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). The *diversity relation* "$\neq$" ([10]) is a binary relation on S, which satisfies the following properties:

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$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z\ .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z)),$$

it called *apartness* (A. Heyting). For subset X of S we say that it is *strongly extensional subset* of S if and only if $x \in X \Longrightarrow (\forall y \in S)(x \neq y \vee y \in X)$. Follows Bridges and Vita's definition, for subsets X and Y of S we say that set X is *set-set apartness* from Y, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \bowtie Y$ instead $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead $\{x\} \bowtie \{y\}$. With $X^C = \{x \in S : x \bowtie X\}$ we denote *apartness complement* of X. For a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ we say that it is a *strongly extensional* function if and only if $(\forall a, b \in S)(f(a) \neq_T f(b) \Longrightarrow a \neq_S b)$.

Semilattice-ordered semigroup is important algebraic structure. It studied, for example, by Martin Kuril and Libor Polka ([6]). J van Plato studied in his article [8] semilattice in Constructive Algebra. In this article we take over from [12] definitions of semilattice-ordered semigroup S (with apartness and strongly extensional internal operation) and anti-ideals. (In that paper is given some examples of semillatice-ordered semigroups.) In this paper we give a natural construction of the positive quasi-antiorder relation σ on S, generated by an anti-ideal K of S.

Following to classical definition in [6], for algebraic structure $((S, =, \neq, \cdot, 1), \otimes)$ is called a (*strong*) *semilattice-ordered semigroup* if :
(i) $(S, =, \neq, \cdot, 1)$ is a semigroup, where the semigroup operation is strongly extensional in the following way

$$(\forall a, b, c \in S)((ac \neq bc \vee ca \neq cb) \Longrightarrow a \neq b);$$

(ii) $(S, =, \neq, \otimes)$ is a semilattice, i.e. $(S, \otimes)$ is a commutative semigroup with $(\forall x \in S)(x \otimes x = x)$ where the semigroup operation is strongly extensional:

$$(\forall a, b, c \in S)((a \otimes c \neq b \otimes c \vee c \otimes a \neq c \otimes b) \Longrightarrow a \neq b);$$

(iii) $(\forall a, b, c \in S)((a(b \otimes c) = ab \otimes ac) \wedge ((a \otimes b)c = ac \otimes bc))$; and
(iv) $(\forall x \in S)(x \otimes 1 = 1)$.

A relation α on S is *antiorder* ([4], [5]) on S if and only if

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1} \text{ (linearity)},$$
$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \Longrightarrow (x, y) \in \alpha) \wedge ((zx, zy) \in \alpha \Longrightarrow (x, y) \in \alpha))$$
and
$$(\forall x, y, z \in S)((x \otimes z, y \otimes z) \in \alpha \Longrightarrow (x, y) \in \alpha).$$

A relation on S is a *quasi-antiorder* ([5], [11]) on S if and only if

$$\sigma(\subseteq \alpha) \subseteq \neq,\ \sigma \subseteq \sigma * \sigma,$$
$$(\forall x, y, z \in S)(((xz, yz) \in \sigma \Longrightarrow (x, y) \in \sigma) \wedge ((zx, zy) \in \sigma \Longrightarrow (x, y) \in \sigma))$$
and
$$(\forall x, y, z \in S)((x \otimes z, y \otimes z) \in \sigma \Longrightarrow (x, y) \in \sigma).$$

In the following lemma we show that semilattice-ordered semigroup is equipped with the natural defined anti-order relation:

Lemma ([12], Lemma 2.2) *If* $(S, =, \neq, , \otimes)$ *is a semilattice-ordered semigroup and we define, for any* a, b *of* S*,*

$$(a, b) \in \alpha \Longleftrightarrow a \otimes b \neq a,$$

then the structure $(S, =, \neq, \cdot, \otimes)$ *is an ordered semigroup under antiorder* α*.*

Proof: (i) It is clear that the relation α is consistent.

(ii) Let a, b, c be arbitrary elements of S such that $(a, c) \in \alpha$, i.e. such that $a \otimes c \neq a$. Then,

$$a \otimes c \neq a \Longrightarrow a \otimes c \neq b \otimes a \ \vee \ b \otimes a \neq a.$$

If $b \otimes a \neq a$, then $(a, b) \in \alpha$. Suppose that $a \otimes c \neq b \otimes a$. Then, $a \otimes c \neq a \otimes b \otimes c$ or $a \otimes b \otimes c \neq b \otimes a$. In the first case, we conclude:

$$\begin{aligned} a \otimes c \neq a \otimes b \otimes c &\Longrightarrow a \neq a \otimes b \ \vee \ c \neq c \\ &\Longrightarrow (a, b) \in \alpha. \end{aligned}$$

In the second case, we have

$$\begin{aligned} a \otimes b \otimes c \neq b \otimes a &\Longrightarrow b \otimes c \neq b \ \vee \ a \neq a \\ &\Longrightarrow (b, c) \in \alpha\ . \end{aligned}$$

Therefore, the relation α is cotransitive.

(iii) Let a and b be arbitrary element of S such that $a \neq b$. Thus, $a \neq a \otimes b$ or $a \otimes b \neq b$. So, we have $a \neq b \Longrightarrow (a, b) \in \alpha \ \vee \ (b, a) \in \alpha$, and the relation α is linear.

(iv) Let a, b, c be arbitrary elements of semigroup $(S, =, \neq, \cdot, \otimes)$ such that $(ac, bc) \in \alpha$. Then,

$$\begin{aligned} ac \otimes bc \neq ac &\Longleftrightarrow (a \otimes b)c \neq ac \\ &\Longrightarrow a \otimes b \neq a \\ &\Longrightarrow (a, b) \in \alpha\ . \end{aligned}$$

Analogously, we derive the implication $(ca, cb) \in \alpha \Longrightarrow (a, b) \in \alpha$.

(v) Let a, b, c be elements of S such that $(a \otimes c, b \otimes c) \in \alpha$, i.e. such that $a \otimes c \otimes b \otimes c \neq a \otimes c$. Thus, $a \otimes b \otimes c \neq a \otimes c$ and $a \otimes b \neq a$. Hence, $(a, b) \in \alpha$.

Finally, the relation α is an antiorder relation on semigroup $(S, =, \neq,)$ and the structure $(S, =, \neq\ , \alpha)$ is a semigroup ordered under antiorder. □

Let $(S,=,\neq,\cdot,1,\otimes)$ be a semilattice-ordered semigroup. A subset K of S is its anti-ideal if and only if
(0) $ab \in K \Longrightarrow a \in K \ \vee \ b \in K$,
(1) $a \otimes b \in K \Longrightarrow a \in K \ \vee \ b \in K$,
(2) $b \in K \Longrightarrow a\alpha b \ \vee \ a \in K$.

Remarks:
1. Any anti-ideal of semilattice ordered semigroup is strongly extensional subset of S. Indeed, if $b \in K$ then by (2) of definition, $a\alpha b \vee a \in K$. Since the relation α is consistent, then $a \neq b$ or $a \in K$. So, the set K is a strongly extensional subset of S.

2. If K is an anti-ideal of a semilattice-ordered semigroup S, then $(\forall a,b \in S)(a \in K \Longrightarrow a \otimes b \in K)$ and $(\forall a,b \in S)(a \in K \wedge a \leq b \Longrightarrow b \in K)$. Let $a \in K$ and b be arbitrary element of S. Then, by (2) of definition of anti-ideal, we have $(a \otimes b)\alpha a$ or $a \otimes b \in K$. Since $\neg((a \otimes b) \otimes a \neq a \otimes b)$, we have to $a \otimes b \in K$. The second implication immediately follows from the first.

3. $a \otimes b \bowtie K \Longrightarrow (a \bowtie K \wedge b \bowtie K)$. In fact, suppose that $a \otimes b \bowtie K$. If t is an arbitrary element of K, then we have $t \neq a \vee a \in K$ and $t \neq b \vee b \in K$. Since in the case $a \in K$ or $b \in K$ we have $a \otimes b \in K$, we conclude $a \bowtie K$ and $b \bowtie K$.

4. Let K be an anti-ideal of semilattice-ordered semigroup $(S,=,\neq,\cdot,\otimes)$. Then sets $\neg K$ and K^C are ideals of S.
Proof: (i) Let $a \in \neg K$ and $b \leq a$. Suppose that $b \in K$. Then, by (2), we have $a\alpha b \vee a \in K$. It is impossible by hypothesis. So, $b \in \neg K$. Let $a \in \neg K$ and $b \in \neg K$. Then, from $a \otimes b \in K$ we conclude that $a \in K$ and $b \in K$. Since it is impossible, we have $a \otimes b \in \neg K$.

(ii) Let $a \in K^C$ and $b \leq a$, and let t be an arbitrary element of K. Then $t \neq b$ or $b \in K$. Since, from $b \in K$ follows $a \in K$. It is a contradiction. So, we have $b \bowtie K$. Let $a \in K^C$ and $b \in K^C$ and let t be an arbitrary element of K. Then, $t \neq a \otimes b$ or $a \otimes b \in K$. Since the second case is impossible, we conclude that $a \otimes b \bowtie K^C$. ♦

For undefined notions and notations of semigroup items we referred to book [2] and articles [6], [8] and of items of Constructive Algebra we referred to books [1], [3], [7] and [13], and to the papers [4], [5], [9]-[12].

2 The Result

A quasi-antiorder σ on a semillatice-ordered semigroup $((S,=,\neq,\cdot,1),\otimes)$ is positive if and only if

$$(\forall a,b \in S)((a,ab) \bowtie \sigma \wedge (a,ba) \bowtie \sigma); \text{ and}$$

$$(\forall a, b \in S)((a, ab \otimes ba) \bowtie \sigma \ \wedge \ (b, ab \otimes ba) \bowtie \sigma)).$$

In the following theorem we show that the semilattice-ordered semigroup is equipped with the natural defined quasi-antiorder relation:

Theorem: *Let K be an anti-ideal of a semilattice-ordered semigroup S. Then the relation σ on S, defined by $(a, b) \in \sigma \Longleftrightarrow (\exists x, y \in S^1)(xby \in K \wedge xay \bowtie K)$ is a positive quasi-antiorder such that $\sigma \subseteq \alpha$.*

Proof: (i) If $(a, b) \in \sigma$, then $(\exists x, y \in S^1)(xby \in K \wedge xay \bowtie K)$ and out of $xby \neq xay$ we got $b \neq a$. So, the relation σ is a consistent relation.

(ii) Let a, b, c arbitrary elements of S such that $(a, c) \in \sigma$. Let u be an arbitrary element of K. Then, by strongly extensionality of K, we conclude: $u\alpha xby \vee xby \in K$. If $u \neq xby$ and $xay \in K$, then $(a, b) \in \sigma$. If $xby \in K$ and $xcy \bowtie K$, then $(b, c) \in \sigma$.

(iii) Let a, b, c be arbitrary elements of S such that $(ac, bc) \in \sigma$, i.e. $(\exists x, y \in S)(xacy \in K \wedge xbcy \bowtie K)$. Then, $(\exists x, cy \in S)(xa(cy) \in K \wedge xb(cy) \bowtie K)$. So, $(a, b) \in \sigma$. Similarly, we have $(ca, cb) \in \sigma \Longrightarrow (a, b) \in \sigma$. Therefore, the relation σ is compatible with the operation "$\cdot$"

(iv) Suppose that a, b, c be arbitrary elements of S such that $(a \otimes c,\ b \otimes c) \in \sigma$. Then, there exist elements x, y of S such that $x(a \otimes c)y \in K \ \wedge\ x(b \otimes c)y \bowtie K$. Then, $xay \otimes xcy \in K$ and $xby \otimes xcy \bowtie K$, and out of this we conclude $(\exists x, y \in S)((xay \in K \vee xcy \in K) \wedge (xby \bowtie K \wedge xcy \bowtie K)$. Thus, $(\exists x, y \in S)(xay \in K \wedge xby \bowtie K)$ because the following $(\exists x, y \in S)(xcy \in K \wedge xcy \bowtie K)$ is impossible. Therefore, $(a, b) \in \sigma$.

(v) If $(a, b) \in \sigma$, ie. if $xay \in K$ and $xby \bowtie K$ for some x, y of S, then $(xay, xby) \in \alpha$ or $xby \in K$. Since the second case is impossible, we have $(xay, xby) \in \alpha$ and $(a, b) \in \alpha$.

(vi) Let a, b be arbitrary elements of S. Then:

$$\begin{aligned}(u, v) \in \sigma &\Longrightarrow (u, a) \in \sigma \vee (a, ab) \in \sigma \vee (ab, v) \in \sigma \\ &\Longrightarrow (u, v) \neq (a, ab) \ \vee\ (a, ab) \in \sigma \subseteq \alpha \\ &\Longrightarrow (a, ab) \neq (u, v) \in \sigma\end{aligned}$$

because out of $(a, ab) \in \sigma \subseteq \alpha$ we have a contradiction. Indeed, suppose that $a \otimes (ab) \neq a$. Thus, $a(1 \otimes b) \neq a$, i.e. $a \neq a$. For the fact $(a, ba) \bowtie \sigma$, we have similar proof.

(vii) Let a, b, u, v be elements of S such that $(u, v) \in \sigma$. Thus, $(a, ab \otimes ba) \in \sigma$ or $(a, ab \otimes ba) \neq (u, v)$. Since the first case is impossible, we have $(a, ab \otimes ba) \bowtie \sigma$. In fact, out of the hypothesis $(a, ab \otimes ba) \in \sigma \subseteq \alpha$ we conclude $a \neq a$. The proof for $(b, ab \otimes ba) \bowtie \sigma$ we got analogously. □

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