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Images, Co-images and Factorization in the Category of G-Sets

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Abstract

In this paper, we study the existence of images and co-images in the category of G-sets. We also show that the category of G-sets is uniquely factorizable.

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1. Introduction

G-sets, as such, arise out of the idea of groups with operators, which has been discussed in [3] and has been used to define orbits of a permutation and to prove Sylow theorems for groups. Recently some interesting properties of G-sets have been studied in [7, 8]. A G-set can be defined as follows: Let G be a group and X be a set. Then X is said to be a G-set if there exists a mapping $\phi : G \times X \to X$ such that (i) $\phi(ab, x) = \phi(a, \phi(b, x))$ for all $a, b \in G$ and $x \in X$ (ii) $\phi(e, x) = x$, where e is the identity of G. The G-set X together with the mapping $\phi : G \times X \to X$ will be denoted by the pair (X, ϕ). To simplify the notations, one can write $\phi(a, x)$ as ax. Furthermore, if X and Y are two G-sets, then a mapping $f : X \to Y$ is said to be a G-morphism if $f(ax) = af(x)$ for all $a \in G$, $x \in X$.

Since the composition of G-morphisms is a G-morphism and every identity mapping is a G-morphism, we can construct a category by taking G-sets as the objects of the category and G-morphisms as the morphisms of the category. We call this category as the category of G-sets and denote it by $\mathcal{G}$-Sets. The development of this paper is based on the detailed knowledge of category theory. For the categorical concepts and techniques one is referred to [1,2,4,5,6].

2. Preliminaries

In this section we include some notions and results on G-sets that are required in the present analysis. For more details, readers are advised to consult [7,8].

Proposition 2.1. Image of a G-morphism is a G-set.

Proof. Let X and Y be any two G-sets and $f : X \to Y$ be a G-morphism. In order to prove that $Im(f) = \{f(x) \mid x \in X\}$ is a G-set, define a mapping $\phi : G \times Im(f) \to Im(f)$ by $\phi(a, f(x)) = f(ax)$ for all $a \in G$, $x \in X$.
Clearly, $f(ax) \in Im(f)$, therefore ϕ is well defined and it can be easily seen that $Im(f)$ is a G-set.

Proposition 2.2. Let $f : X \to Y$ be a G-morphism. Then for any $x_1, x_2 \in X$, a relation $\sim_f$ on X defined by $x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2)$ is an equivalence relation.

Proof. For any $x \in X$, we have $f(x) = f(x)$ implying thereby $x \sim_f x$. Now, suppose $x_1 \sim_f x_2$. Then $f(x_1) = f(x_2)$. So, we have $f(x_2) = f(x_1)$ implies that $x_2 \sim_f x_1$. Further, suppose $x_1 \sim_f x_2$ and $x_2 \sim_f x_3$. Then $f(x_1) = f(x_2)$ and $f(x_2) = f(x_3)$. Thus, we have $f(x_1) = f(x_2) = f(x_3)$ implying thereby $x_1 \sim_f x_3$. Consequently, $\sim_f$ is an equivalence relation.

Proposition 2.3. Let $f : X \to Y$ be a G-morphism. Then for any $x \in X$, the equivalence class of x under the equivalence relation $\sim_f$ is a G-set.

Proof. Let $[x]$ be an equivalence class of x with respect to $\sim_f$. Define a mapping $\phi : G \times [x] \to [x]$ by $\phi(g, x') = x'$ for all $g \in G, x' \in [x]$. Trivially, $([x], \phi)$ is a G-set.

We shall denote by $X/{\sim_f}$ the set of all equivalence classes with respect to $\sim_f$.

Proposition 2.4. The set $X/{\sim_f}$ is a G-set.

Proof. Let X be a G-set and $\sim_f$ be an equivalence relation on X. Define a mapping $\phi : G \times X/{\sim_f} \to X/{\sim_f}$ by $\phi(g, [x]) = [gx]$ for all $g \in G, x \in X$.

To show that ϕ is well defined. Suppose, $[x_1] = [x_2]$ which implies $x_1 \sim_f x_2$ yielding thereby $f(x_1) = f(x_2)$. Thus, we have $f(gx_1) = f(gx_2)$ implying $[gx_1] = [gx_2]$ and hence $\phi(g, [x_1]) = \phi(g, [x_2])$. Therefore ϕ is well defined.

For any $a, b \in G$ and $x \in X$, we get $\phi(ab, [x]) = \phi(a, [bx]) = \phi(a, \phi(b, [x]))$. Again, if e is the identity element of G and $x \in X$, then $\phi(e, [x]) = [ex] = [x]$, which shows that $(X/\sim_f, \phi)$ is a G-set.

Definition 2.1. Let X be a G-set. Then a G-subset S of X together with an inclusion morphism $i : S \to X$ is called a sub-object of X in $\mathcal{G}$-Sets.

Definition 2.2. Let X be a G-set and let $\sim_f$ be a G-equivalence relation on X. Then the quotient set $X/\sim_f$ together with natural projection $p : X \to X/\sim_f$ is called a quotient object of X in $\mathcal{G}$-Sets.

3. Main Results

Theorem 3.1. The category $\mathcal{G}$-Sets has images.

Proof. Let $f : X \to Y$ be a morphism in the category $\mathcal{G}$-Sets. Consider the set $I = \{f(x) \mid x \in X\}$, then in view of Proposition 2.1, I is a G-set. Define an inclusion morphism $u : I \to Y$ by $u(f(x)) = f(x)$ for all $f(x) \in I$, then u is a monomorphism in $\mathcal{G}$-Sets. We claim that $u : I \to Y$ is the image of f.

Consider the morphism $\alpha : X \to I$ defined by $\alpha(x) = f(x)$ for all $x \in X$. For any $x \in X$, we get $(u \circ \alpha)(x) = \alpha(x) = f(x)$ which implies $f = u \circ \alpha$.

Let $u' : S \to Y$ be a sub-object of Y and $\alpha' : X \to S$ be a morphism in $\mathcal{G}$-Sets such that $u' \circ \alpha' = f$. Define a morphism $\eta : I \to S$ by $\eta(f(x)) = \alpha'(x)$ for all $x \in X$. For any $x_1, x_2 \in X$, let $f(x_1) = f(x_2)$ implying $u'(\alpha'(x_1)) = u'(\alpha'(x_2))$. Thus, we get $\alpha'(x_1) = \alpha'(x_2)$, as u' is a monomorphism. Henceforth, the mapping η is well defined. The mapping η is a G-morphism because for any $a \in G$, one gets $\eta(af(x)) = \eta(f(ax)) = \alpha'(ax) = a(\alpha'(x)) = a(\eta(f(x)))$.

Moreover, for any $f(x) \in I$, we get $(u' \circ \eta)(f(x)) = u'(\eta(f(x))) = u'(\alpha'(x)) = f(x) = u(f(x))$. Therefore, we get $u' \circ \eta = u$.

Lastly, we show that η is unique. Suppose there exists another morphism $\xi : I \to S$ in $\mathcal{G}$-Sets such that $\xi \circ \alpha = \alpha'$. Then for any $x \in X$, we have $(\xi \circ \alpha)(x) = \alpha'(x)$ implying $\xi(\alpha(x)) = \alpha'(x)$. Thus, we have $\xi(f(x)) = \alpha'(x) = \eta(f(x))$ for all $f(x) \in I$. Therefore, $\xi = \eta$ and hence the result follows.

Proposition 3.1. The category $\mathcal{G}$-Sets has epimorphic images.

Proof. In the above Theorem 3.1, the morphism α is an epimorphism, therefore the category $\mathcal{G}$-Sets has epimorphic images [5].

Theorem 3.2. The category $\mathcal{G}$-Sets has co-images.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{G}$-Sets and $\sim_f$ be an equivalence relation on X. Then by Proposition 2.4, $X/{\sim_f}$ is a G-set. Define a natural morphism $p : X \to X/{\sim_f}$ by $p(x) = [x]$ for all $x \in X$, then p is an epimorphism in $\mathcal{G}$-Sets. We claim that $p : X \to X/{\sim_f}$ is the co-image of f.

Consider a morphism $j : X/{\sim_f} \to Y$ defined by $j([x]) = f(x)$ for all $x \in X$, then j is a monomorphism in $\mathcal{G}$-Sets. For any $x \in X$, we get $(j \circ p)(x) = j([x]) = f(x)$ which gives $j \circ p = f$.

Let $p' : X \to Q$ be a quotient object of X and $j' : Q \to Y$ be a morphism in $\mathcal{G}$-Sets such that $j' \circ p' = f$. Define a morphism $\eta : Q \to X/{\sim_f}$ by $\eta(p'(x)) = [x]$ for all $x \in X$. For any $x_1, x_2 \in X$, let $p'(x_1) = p'(x_2)$ yielding thereby $j'(p'(x_1)) = j'(p'(x_2))$ which implies $f(x_1) = f(x_2)$ giving $[x_1] = [x_2]$. Therefore, the mapping η is well defined. For any $a \in G$, one gets $\eta(ap'(x)) = \eta(p'(ax)) = [ax] = a(\eta(p'(x)))$. Thus, η is a morphism in $\mathcal{G}$-Sets.

Moreover, for any $x \in X$, we have $(\eta \circ p')(x) = [x] = p(x)$. Henceforth, $\eta \circ p' = p$.

Finally, we show that η is unique. Suppose there exists another morphism $\xi : Q \to X/{\sim_f}$ in $\mathcal{G}$-Sets such that $\xi \circ p' = p$. Then for any $x \in X$, we have $(\xi \circ p')(x) = p(x)$ implying thereby $\xi(p'(x)) = [x] = \eta(p'(x))$ for all $p'(x) \in Q$. Thus, we get $\xi = \eta$ and hence the result follows.

Theorem 3.3. The category $\mathcal{G}$-Sets is uniquely factorizable.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{G}$-Sets and $\sim_f$ be an equivalence relation on X. Then by Proposition 2.4, $X/{\sim_f}$ is a G-set.

Define a mapping $q : X/{\sim_f} \to Y$ by $q([x]) = f(x)$ for all $x \in X$. For any $x_1, x_2 \in X$, we have $[x_1] = [x_2]$ implying $x_1 \sim_f x_2$. Thus, we have $f(x_1) = f(x_2)$ and henceforth the mapping q is well defined. Obviously, q is a monomorphism in $\mathcal{G}$-Sets.

Define a natural morphism $p : X \to X/\sim_f$ by $p(x) = [x]$ for all $x \in X$. Trivially, p is an epimorphism in $\mathcal{G}$-Sets. Thus the morphism f can be factorized as $X \xrightarrow{f} Y = X \xrightarrow{p} X/\sim_f \xrightarrow{q} Y$, where q is a monomorphism and p is an epimorphism in $\mathcal{G}$-Sets.

To show that f is uniquely factorizable, let us consider another factorization of f as $X \xrightarrow{f} Y = X \xrightarrow{\alpha} B \xrightarrow{\beta} Y$, where $\alpha : X \to B$ is an epimorphism and $\beta : B \to Y$ is a monomorphism in $\mathcal{G}$-Sets.

Consider a mapping $\eta : X/\sim_f \to B$ defined by $\eta([x]) = \alpha(x)$ for all $x \in X$. For any $x_1, x_2 \in X$, suppose $[x_1] = [x_2]$, then $f(x_1) = f(x_2)$. Thus, we have $\beta(\alpha(x_1)) = \beta(\alpha(x_2))$ which yields $\alpha(x_1) = \alpha(x_2)$, as β is a monomorphism. Therefore, the mapping η is well defined.

For any $a \in G$, we get $\eta(a[x]) = \eta([ax]) = \alpha(ax) = a(\alpha(x)) = a(\eta([x]))$ which amounts to say that η is a morphism in $\mathcal{G}$-Sets. Moreover, for any $x \in X$, one gets $(\eta \circ p)(x) = \eta(p(x)) = \eta([x]) = \alpha(x)$ which yields $\eta \circ p = \alpha$. Also, for any $[x] \in X/\sim_f$, we get $(\beta \circ \eta)([x]) = \beta(\alpha(x)) = f(x) = q([x])$ which implies $\beta \circ \eta = q$.

To show that η is unique, suppose there exists another $\xi : X/\sim_f \to B$ in $\mathcal{G}$-Sets such that $\xi \circ p = \alpha$. Then for any $x \in X$, we have $(\xi \circ p)(x) = \alpha(x)$ yielding $\xi([x]) = \alpha(x) = \eta([x])$ for all $[x] \in X/\sim_f$ which gives $\xi = \eta$.

Finally, we show that η is an isomorphism.

We have, $\eta \circ p = \alpha$ yielding thereby $\eta \circ p$ is an epimorphism as α is an epimorphism which amounts to say that η is an epimorphism.

Also, we have $f = \beta \circ \alpha = (\beta \circ \eta) \circ p$. But $f = q \circ p$, therefore we get $q \circ p = (\beta \circ \eta) \circ p$ which gives $q = \beta \circ \eta$ as p is an epimorphism. Thus, $\beta \circ \eta$ is a monomorphism as q is a monomorphism. Therefore, η is a monomorphism. Consequently, the category $\mathcal{G}$-Sets is uniquely factorizable.

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