International Mathematical Forum, 4, 2009, no. 8, 383 - 393

On the Category of G-sets-I

S. M. A. Zaidi, M. Irfan and Gulam Muhiuddin

Department of Mathematics
Aligarh Muslim University
Aligarh - 202 002, India
zaidimath@rediffmail.com
mohammad_irfanamu@yahoo.com
gmchishty@math.com

Abstract

In this paper, we study the behaviour of monomorphisms, epimorphisms, co-retractions and retractions in the category of G-sets. Further, some special objects are discussed here and obtain that the category of G-sets is balanced, well powered and co-well powered.

Mathematics Subject Classification: 18A05, 20C05

Keywords: G-sets, G-equivalence relation and G-morphisms

1. Introduction

Categories, functors and natural transformations were introduced by Eilenberg and MacLane [4] in 1945. But, in this paper it was not clear that the concepts of category theory would be more than a convenient language and so it remained for approximately fifteen years. In the sixties, Lawvere outlined basic framework for the development of an entirely original approach to logic and the foundations of mathematics; he proposed an axiomatization of the category of categories (Lawvere 1966), an axiomatization of the category of sets (Lawvere 1964), characterized cartesian closed categories and showed their connections to logical systems and various logical paradoxes (Lawvere 1969). The 1970s saw the development and application of the concept in many different directions.

Finally, from 1980 to this day, category theory occupies a central position not only in contemporary mathematics but also in theoretical computer science where it has firm roots and contributes, among other things, to the development of the semantics of programming and the development of new logical systems ([10], [11], [13], [14]). On the other hand, its applications to mathematics are becoming more diversified and it even touches upon theoretical physics where higher-dimensional category theory, which is to category theory what higher-dimensional geometry is to plane geometry, is used in the study of the so-called "quantum groups", or in quantum field theory [2].

Category theory can roughly be described as a general mathematical theory of structures and systems of structures. It is at the very least, a very powerful language or conceptual framework which allows us to see, among other things, how structures of different kinds are related to one another as well as the universal components of a family of structures of a given kind.

The theory of groups first dealt with permutation groups. Later, the notion of an abstract group was introduced in order to examine properties of permutation groups which did not refer to the set on which the permutations acted. However, one is primarily interested in permutation groups in geometry. Also, permutation groups are used in counting techniques that are important in finite group theory. In fact, G-set is an extension of a permutation on a set to a group action on a set. G-sets, as such, arise out of the idea of groups with operators which has been discussed in [9] and has been used to define orbits of a permutation and to prove Sylow theorems for groups.

In this paper, we study some properties of a structural category i.e., a category of G-sets. To avoid unneccesary bulk, we shall give here only some basic concepts of G-sets. For the categorical concepts and techniques, the readers are advised to consult [1, 5, 6, 7, 8, 15].

2. Preliminaries

In this section, we recall the following aspects of G-sets from [15, 16, 17] that will be needed in the sequel:

Definition 2.1. Let G be a group and X be a set. Then X is said to be a G-set (or a set with operator G) if there exists a mapping $\phi : G \times X \to X$ such that for all $a, b \in G$ and $x \in X$ the following conditions are satisfied:

(i) $\phi(ab, x) = \phi(a, \phi(b, x))$,

(ii) $\phi(e, x) = x$,

where e is the identity of G. The G-set X defined above will be denoted by the pair (X, ϕ).

For the sake of convenience, one can denote $\phi(a, x)$ as ax. Under this notation, above conditions become

(i) $(ab)x = a(bx)$,

(ii) $ex = x$.

Definition 2.2. Let (X, ϕ) is a G-set. Then a subset A of X is called a G-subset of X if (A, ϕ) is also a G-set.

Definition 2.3. Let X and Y be two G-sets. Then a mapping $f : X \to Y$ is called a G-morphism from X to Y if

$$f(ax) = af(x) \ \text{ for all } a \in G, \ x \in X.$$

Remark 2.1.

(i) For any singleton set $\{w\}$, we can define a mapping $\phi : G \times \{w\} \to \{w\}$ by $\phi(g, w) = w$ for all $g \in G$, $w \in \{w\}$. Trivially, $(\{w\}, \phi)$ is a G-set. Therefore, every singleton set is a G-set.

(ii) The null set $\emptyset$ is a G-set.

(iii) Every identity mapping is a G-morphism.

(iv) Composition of G-morphisms is again a G-morphism.

(v) Image of a G-morphism is a G-set.

Example 2.1. The set $\{0, 1\}$ is a G-set under the mapping $\phi : G \times \{0, 1\} \to \{0, 1\}$ defined by

$$\phi(g, x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x = 1, \end{cases}$$

for all $g \in G$, $x \in \{0, 1\}$.

We shall use the following lemmas in our main results:

Lemma 2.1. Let X and Y be two G-sets. Then

(i) $X \cap Y$ is a G-set,

(ii) $X \times Y$ is a G-set,

(iii) Disjoint union of X and Y is a G-set.

Lemma 2.2. Let (X, ϕ) be a G-set. Then

(i) for any $x, y \in X$, a relation $\sim_G$ on X defined by $x \sim_G y \Leftrightarrow \ y = \phi(g, x)$ for some $g \in G$, is an equivalence relation,

(ii) the set $X/\sim_G$ of all G-equivalence classes is a G-set.

3. Main Results

Since the composition of G-morphisms is again a G-morphism and every identity mapping is a G-morphism, therefore we can construct a category by taking the class of all G-sets as the class of objects of the category and the class of all G-morphisms as the class of morphisms of the category. We call this category as the category of G-sets and denote it by $\mathcal{G}$-Sets.

Theorem 3.1. A morphism $\alpha : X \to Y$ in $\mathcal{G}$-Sets is injective if and only if it is left cancellable.

Proof. Let $\alpha : X \to Y$ be an injective morphism in $\mathcal{G}$-Sets. For any $Z \in \mathcal{G}$-Sets, suppose there are two morphisms $h, k : Z \to X$ in $\mathcal{G}$-Sets such that $\alpha \circ h = \alpha \circ k$. Then for any $z \in Z$, we have

$$(\alpha \circ h)(z) = (\alpha \circ k)(z)$$

$$\Longrightarrow \qquad \alpha(h(z)) = \alpha(k(z))$$

$$\Longrightarrow \qquad h(z) = k(z) \qquad (\text{ as } \alpha \text{ is injective})$$

$$\Longrightarrow \qquad h = k$$

implying thereby α is left cancellable.

Conversely, suppose that α is left cancellable and let $\alpha(x_1) = \alpha(x_2)$ for $x_1, x_2 \in X$. Consider the set $C = \{w\}$ and define two morphisms $h, k : C \to X$ such that $h(w) = x_1$ and $k(w) = x_2$. Obviously, h and k are G-morphisms. Then we have

$$C \xrightarrow{h} X \xrightarrow{\alpha} Y = C \xrightarrow{k} X \xrightarrow{\alpha} Y$$

i.e.,

$$\alpha \circ h = \alpha \circ k$$

$$\Longrightarrow \qquad h = k \qquad (\text{ as } \alpha \text{ is left cancellable})$$

$$\Longrightarrow \qquad h(w) = k(w) \text{ for all } w \in C$$

which yields $x_1 = x_2$ implying thereby α is injective. This completes the proof.

Following result is a direct consequence of above theorem:

Corollary 3.1. A morphism in the category $\mathcal{G}$-Sets is a monomorphism if and only if it is an injective.

Theorem 3.2. A morphism $\alpha : X \to Y$ in $\mathcal{G}$-Sets is surjective if and only if it is right cancellable.

Proof. Let us consider $\alpha : X \to Y$ be a surjective morphism in $\mathcal{G}$-Sets. For any $Z \in \mathcal{G}$-Sets, suppose there are two morphisms $h, k : Y \to Z$ in $\mathcal{G}$-Sets such that $h \circ \alpha = k \circ \alpha$. Since α is surjective, for every $y \in Y$ there exists $x \in X$ such that $y = \alpha(x)$. Then we have

$$\begin{aligned} h(y) &= h(\alpha(x)) \\ &= (h \circ \alpha)(x) \\ &= (k \circ \alpha)(x) \\ &= k(\alpha(x)) \\ &= k(y) \quad \text{for all } y \in Y \end{aligned}$$

implying thereby $h = k$ which amounts to say that α is right cancellable.

Conversely, suppose that α is right cancellable. Define two morphisms $h, k : Y \to \{0, 1\} \cup Im(\alpha)$ such that

$$h(y) = \begin{cases} y & \text{if } \ y \in Im(\alpha) \\ 0 & \text{otherwise,} \end{cases}$$

$$k(y) = \begin{cases} y & \text{if } \ y \in Im(\alpha) \\ 1 & \text{otherwise.} \end{cases}$$

By Example 2.1, $\{0, 1\}$ is a G-set and by Remark 2.1(v), $Im(\alpha)$ is also G-set, therefore, in view of Lemma 2.1(iii), $\{0, 1\} \cup Im(\alpha)$ is a G-set.

In order to prove that h and k are G-morphisms we show that if $y \in Im(\alpha)$, then $ay \in Im(\alpha)$ for all $a \in G$. To substantiate this, let $y \in Im(\alpha)$, then

$$\alpha(x) = y \quad \text{for some } x \in X$$

$$\Longrightarrow \qquad a\alpha(x) = ay$$

$$\Longrightarrow \qquad \alpha(ax) = ay$$

which yields $ay \in Im(\alpha)$. Therefore, we have

$$h(ay) = ay$$
$$= a(h(y)).$$

Again, we show that if $y \notin Im(\alpha)$, then $ay \notin Im(\alpha)$ for all $a \in G$. Suppose, on contrary that $y \notin Im(\alpha)$ which amounts to say that $ay \in Im(\alpha)$, then

$$\alpha(x) = ay \quad \text{for some } x \in X$$
$$\Longrightarrow \qquad a^{-1}(\alpha(x)) = y$$
$$\Longrightarrow \qquad \alpha(a^{-1}x) = y$$

implying thereby $y \in Im(\alpha)$ which is a contradiction.
Thus, we have

$$h(ay) = 0$$
$$= a0 \qquad \text{(by Example 2.1)}$$
$$= a(h(y)).$$

Therefore, h is a G-morphism. Similarly, we can show that k is also a G-morphism. Hence $h, k \in \mathcal{G}$-Sets.
Now, for any $x \in X$, we have

$$(h \circ \alpha)(x) = h(\alpha(x))$$
$$= \alpha(x) \qquad \text{(by definition of } h)$$
$$= k(\alpha(x)) \qquad \text{(by definition of } k)$$
$$= (k \circ \alpha)(x)$$

implies $h \circ \alpha = k \circ \alpha$ which gives $h = k$ (as α is right cancellable). Suppose $\alpha : X \to Y$ is not surjective, then there exists some $y \in Y$ such that $y \notin Im(\alpha)$. Thus, we have $0 = h(y) = k(y) = 1$ which is a contradiction and hence the result follows.

Corollary 3.2. A morphism in the category $\mathcal{G}$-Sets is an epimorphism if and only if it is surjective.

Definition 3.1. A morphism $\alpha : X \to Y$ in $\mathcal{G}$-Sets is called co-retraction(section) if and only if there exists a morphism $\beta : Y \to X$ in $\mathcal{G}$-Sets such that $\beta \circ \alpha = I_X$.

Dually, we have the following definition:

Definition 3.2. A morphism $\alpha : X \to Y$ in $\mathcal{G}$-Sets is called retraction if and only if there exists a morphism $\beta : Y \to X$ in $\mathcal{G}$-Sets such that $\alpha \circ \beta = I_Y$.

Theorem 3.3. A morphism $\alpha : X \to Y$ in $\mathcal{G}$-Sets is a monomorphism if and only if it is a section (co-retraction).

Proof. Let X be a G-set together with a fixed element $w \in X$ such that $aw = w$ for all $a \in G$ and let $\alpha : X \to Y$ be a monomorphism in $\mathcal{G}$-Sets. For any $y \in Y$, define a mapping $\beta : Y \to X$ by

$$\beta(y) = \begin{cases} x & \text{if} \quad y \in Im(\alpha) \text{ and } \alpha(x) = y \ \text{ for some } x \in X \\ \\ w & \text{otherwise.} \end{cases}$$

To show that β is well defined, suppose $y = y'$ for all $y, y' \in Y$. Then either both $y, y' \in Im(\alpha)$ or $y, y' \notin Im(\alpha)$. If $y, y' \notin Im(\alpha)$, then $\beta(y) = w = \beta(y')$.

Suppose, $y, y' \in Im(\alpha)$, then there exist unique $x, x' \in X$ such that $\alpha(x) = y$ and $\alpha(x') = y'$ implying thereby $\beta(y) = x$ and $\beta(y') = x'$. Then β is well defined, for if

$$y = y'$$

$$\Longrightarrow \quad \alpha(x) = \alpha(x')$$

$$\Longrightarrow \quad x = x' \qquad (\text{ as } \alpha \text{ is injective})$$

$$\Longrightarrow \quad \beta(y) = \beta(y').$$

In order to prove that β is a G-morphism, we show that if $y \notin Im(\alpha)$, then $ay \notin Im(\alpha)$ for all $a \in G$. Suppose on contrary that $y \notin Im(\alpha)$ implying thereby $ay \in Im(\alpha)$ which in turn yields

$$\alpha(x) = ay \quad \text{for some } x \in X$$

$$\Longrightarrow \quad a^{-1}(\alpha(x)) = y$$

$$\Longrightarrow \quad \alpha(a^{-1}x) = y$$

yielding thereby $y \in Im(\alpha)$ which is a contradiction.

Therefore, we have

$$\begin{aligned}\beta(ay) &= w \\ &= aw \\ &= a(\beta(y)) \ \text{ for all } y \notin Im(\alpha).\end{aligned}$$

Again we show that if $y \in Im(\alpha)$, then $ay \in Im(\alpha)$ for all $a \in G$. If $y \in Im(\alpha)$, then

$$\alpha(x) = y \quad \text{for some } x \in X$$

$$\Longrightarrow \quad a(\alpha(x)) = ay$$

$$\Longrightarrow \quad \alpha(ax) = ay \ \text{ for all } ay \in Im(\alpha)$$

which implies $ay \in Im(\alpha)$.

Now, if $y \in Im(\alpha)$, then we have

$$\alpha(x) = y \quad \text{for some } x \in X$$

$$\Longrightarrow \quad a(\alpha(x)) = ay$$

$$\Longrightarrow \quad \alpha(ax) = ay \ \text{ for all } ay \in Im(\alpha)$$

implying thereby $\beta(ay) = ax = a(\beta(y))$ which amounts to say that β is a G-morphism.

Finally, we show that $\beta \circ \alpha = I_X$.

Let $x' \in X$ and $\alpha(x') = y'$ for some $y' \in Y$. Then $\beta(y') = x'$ by definition of β. Thus, we have

$$\begin{aligned}(\beta \circ \alpha)(x') &= \beta(\alpha(x')) \\ &= \beta(y') \\ &= x' \\ &= I_X(x') \ \text{ for all } x' \in X\end{aligned}$$

which gives $\beta \circ \alpha = I_X$ and henceforth α is a section.

Conversely, suppose that $\alpha : X \to Y$ is a section, then there exists a morphism $\beta : Y \to X$ such that $\beta \circ \alpha = I_X$ which amounts to say that α is injective and so by Corollary 3.1, it is a monomorphism. This completes the proof.

Theorem 3.4. A morphism $\alpha : X \to Y$ in $\mathcal{G}$-Sets is an epimorphism if and only if it is a retraction.

Proof. Let $\alpha : X \to Y$ be an epimorphism in the category $\mathcal{G}$-Sets. Then for every $y \in Y$ there exists $x \in X$ such that $\alpha(x) = y$. For each $y \in Y$ choose by the axiom of choice and fix such an element x, say x_y, where $x_y \in \alpha^{-1}(y)$. Therefore, we can define a mapping $q : Y \to X$ by $q(y) = x_y$ for all $y \in Y$.

We show that q is a G-morphism.

Since, $$x_y \in \alpha^{-1}(y)$$

$\Longrightarrow$ $$\alpha(x_y) = y$$

$\Longrightarrow$ $$a\alpha(x_y) = ay \quad \text{for all } a \in G$$

$\Longrightarrow$ $$\alpha(ax_y) = ay$$

$\Longrightarrow$ $$ax_y \in \alpha^{-1}(ay)$$

$\Longrightarrow$ $$q(ay) = ax_y$$

$\Longrightarrow$ $$q(ay) = a(q(y))$$

which shows that q is a morphism in $\mathcal{G}$-Sets.
Now, for any $y \in Y$, we have

$$\begin{aligned}(\alpha \circ q)(y) &= \alpha(q(y)) \\ &= \alpha(x_y) \\ &= y\end{aligned}$$

which implies $\alpha \circ q = I_Y$ and so α is a retraction.

Conversely, suppose that $\alpha : X \to Y$ is a retraction, then there exists a morphism $\beta : Y \to X$ such that $\alpha \circ \beta = I_Y$ which amounts to say that α is surjective and so by Corollary 3.2, it is an epimorphism. This completes the proof.

Theorem 3.5. The category $\mathcal{G}$-Sets is balanced.

Proof. Since in the category $\mathcal{G}$-Sets every bimorphism is an isomorphism, therefore the category $\mathcal{G}$-Sets is balanced.

Now, for our further discussion we need the following:

Definition 3.3. Let X be a G-set. Then, a G-subset S of X together with an inclusion morphism $i : S \to X$ is called the sub-object of X in $\mathcal{G}$-Sets.

Definition 3.4. Let X be a G-set and let $\sim_G$ be a G-equivalence relation on X. Then, the quotient set $X/{\sim_G}$ together with natural projection $p : X \to X/{\sim_G}$ is called a quotient object of X in $\mathcal{G}$-Sets.

Trivially, $\emptyset$ forms a G-set i.e., $\emptyset \in \mathcal{G}$-Sets and for any other object $X \in \mathcal{G}$-Sets, there is only one morphism from $\emptyset$ to X with no assignment i.e., $Hom(\emptyset, X)$ is singleton. Thus, we have the following proposition:

Proposition 3.1. The category $\mathcal{G}$-Sets has initial object.

Further, by Remark 2.1(i), every singleton set $\{w\}$ forms a G-set i.e., $\{w\} \in \mathcal{G}$-Sets and for any other object $X \in \mathcal{G}$-Sets, there is only one morphism from X to $\{w\}$ i.e., $Hom(X, \{w\})$ is singleton. Thus, we have the following proposition:

Proposition 3.2. The category $\mathcal{G}$-Sets has terminal object.

Remark 3.5. The category $\mathcal{G}$-Sets has no zero object.

Proposition 3.3. The category $\mathcal{G}$-Sets is well powered.

Proof. Since for any G-set X, the collection of all sub-objects of X is equivalent to the collection of all subsets $P(X)$, the power set of X. But $P(X)$ is a set. Hence, $\mathcal{G}$-Sets is well powered.

Dually, we have

Proposition 3.4. The category $\mathcal{G}$-Sets is co-well powered.

References

[1] T. S. Blyth, *Categories*, Longman, New York 1987.

[2] J. Baez and J. Dolan, *From finite sets to Feynman diagrams*, Mathematics Unlimited-2001 and Beyond, Springer (2001), 29-50.

[3] C.W. Curtic and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol. I, A Wiley Interscience Publication, New York 1981.

[4] S. Eilenberg and S. MacLane, *General theory of natural equivalence* , Trans. Amer. Math. Soc. 58 (1945), 231-294.

[5] P. Freyd, *Abelian Categories*, Happer and Row, New York 1964.

[6] N. Jacobson, *Lectures in Abstract Algebra*, Vol. I, D. Van Nostrand Company, Incorporation, New York 1965.

[7] S. MacLane, *Categories for Working Mathematicians*, Graduate Text in Mathematics, Springer 1972.

[8] B. Mitchell, *Theory of Categories*, New York Academic Press, New York 1965.

[9] A.M. Pitts, *Categorical logic*, Handbook of Logic in Computer Science, Oxford University Press, Oxford (2000), 39-128.

[10] B. Plotkin, *Algebra, categories and databases*, Handbook of Algebra, Elsevier (2000), 79-148.

[11] H. Schubert, *Categories*, Springer-Verlag, New York 1972.

[12] P.J. Scott, *Some aspects of categories in computer science*, Handbook of Algebra, North Holland (2000), 3-77.

[13] R.F.C. Walter, *Categories and Computer Science*, Cambridge University Press 1991.

[14] S.M.A. Zaidi, *Some Problems in Category Theory*, Ph.D. Dissertation, A.M.U., Aligarh 1979.

[15] S.M.A. Zaidi, M. Irfan, S. Khan and Gulam Muhiuddin, *A study on G-sets*, South East Asian J. Math. Math. Sci., 5 (2007)(To appear).

[16] S.M.A. Zaidi, M. Irfan and Gulam Muhiuddin, *A study of G-morphisms* (Communicated).

[17] S. M. A. Zaidi, M. Irfan and Gulam Muhiuddin, *On G-sets* (Communicated).

Received: May 18, 2007