International Mathematical Forum, 4, 2009, no. 8, 377 - 381

On Amenability of Certain Semigroup Algebras

B. Mohammadzadeh

Department of Mathematics
Babol University of Technology, Babol, Mazanderan, Iran
B.Mohammadzadeh@nit.ac.ir

A. Yousofzadeh[1]

Department of Mathematics, Islamic Azad University
Mobareke branch, Mobareke, Isfahan, Iran
a.yousofzade@math.ui.ac.ir

Abstract

In this paper, we study weak amenability and amenability of certain semigroup algebras of foundation topological semigroups.

Mathematics Subject Classification: Primary 43A20; Secondary 46H20

Keywords: Amenability; foundation semigroup; derivation; semigroup algebras; weak amenability

1 Introduction

Throughout, S denotes a locally compact, Hausdroff topological semigroup, and $M(S)$ denotes the space of all bounded complex regular measures on S. This space with the convolution product $*$, and norm $\|\mu\| = |\mu|(S)$ is a Banach algebra. The space of all measures $\mu \in M(S)$ for which the mappings $x \mapsto \delta_x * \mid \mu \mid$ and $x \mapsto \mid \mu \mid * \delta_x$ from S into $M(S)$ are weakly continuous is denoted by $M_a(S)$ (or $\widetilde{L}(S)$) as in [1], where δ_x denotes the Dirac measure at x. Note that the measure algebra $M_a(S)$ defines a two-sided closed L-ideal of $M(S)$ (see [1]). Denote by $L^\infty(S, M_a(S))$ the set of all complex-valued

[1]This paper is stracted from a research supported by Islamic Azad university, Mobareke branch.

bounded functions g on S that are $M_a(S)$-measurable; we identify functions in $L^\infty(S, M_a(S))$ that agree μ-almost everywhere for all $\mu \in M_a(S)$.

Note that in the case where S is discrete (resp. a locally compact group), $L^\infty(S, M_a(S))$ is equal to $\ell^\infty(S)$ (resp. $L^\infty(S)$). Observe that $L^\infty(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra with identity 1, where 1 denotes the constant function one on S and $\|.\|_\infty$ is defined by

$$\| g \|_\infty = \sup\{ \|g\|_{\infty,|\mu|} : \mu \in M_a(S) \} \quad (g \in L^\infty(S, M_a(S))),$$

where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. The equation

$$\tau(g)(\mu) := \mu(g) = \int_S g \, d\mu$$

defines a linear map τ of $L^\infty(S, M_a(S))$ into the continuous dual space $M_a(S)^*$ of $M_a(S)$. Recall from [3] that a semigroups S is called a *foundation semigroup*; if $\cup\{\text{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S. Note that in the case where S is a foundation semigroup with identity, for every $\mu \in M_a(S)$ both mappings $x \mapsto \delta_x * \mid \mu \mid$ and $x \mapsto \mid \mu \mid * \delta_x$ from S into $M_a(S)$ are norm continuous and τ is onto(see [10]). The second dual $M_a(S)^{**}$ of $M_a(S)$ is a Banach algebra with the first Arens product defined by the equations

$$\langle F \odot H, f\rangle = \langle F, Hf\rangle, \quad \langle Hf, \mu\rangle = \langle H, \mu \circ f\rangle$$

for all $F, H \in M_a(S)^{**}$, $f \in M_a(S)^*$, and $\mu \in M_a(S)$.
Let A be Banach algebra, and let X be a Banach A-bimodule. The space of continuous derivations from A into X is denoted by $Z^1(A, X)$, and the space of continuous inner derivations from A into X is denoted by $B^1(A, X)$. The first *cohomology group* of A with coefficients in X is defined to be the linear space $H^1(A, X) = Z^1(A, X)/B^1(A, X)$. Thus $H^1(A, X) = \{0\}$ if and only if each continuous derivation from A into X is inner. Recall that if A be a Banach algebra, then its dual A^* can be a Banach $A-$bimodule, with module actions defined by $(f.a)(b) = f(ab)$ and $(a.f)(b) = f(ba)$ for all $a, b \in A$ and $f \in A^*$. A Banach algebra A is called *weak amenable* if $H^1(A, A^*) = \{0\}$. Also A is called *amenable* if $H^1(A, X^*) = \{0\}$ for all Banach A-bimodule X.

In this paper, we study weak amenability and amenability of $M(S)$ and its closed ideals; in particular $M_a(S)$ of certain foundation semigroups S.

2 The results

It is well-known that for any locally compact group G the group algebra $L^1(G)$ is weakly amenable. The following example shows that this result is not true even for a subsemigroup of a discrete group. We will need the following well-known result.

Proposition 2.1 *If a Banach algebra A is such that $\overline{A^2} \neq A$ then A is not weakly amenable. In particular if $S^2 \neq S$ for a discrete semigroup S then $\ell^1(S)$ is not weakly amenable.*

Example 2.2 Let $S = (\mathbb{N}, +)$. We have $\ell^1(\mathbb{N})$ is not weakly amenable. In fact; since $\mathbb{N}^2 = \mathbb{N} + \mathbb{N} \neq \mathbb{N}$, then by Proposition 2.1, $\ell^1(\mathbb{N})$ is not weakly amenable.

For a locally compact semigroup S, let $M_0(S) = \{\mu \in M(S) : \mu(S) = 0\}$ be the augmentation of $M(S)$. It is immediate that $M_0(S)$ is indeed an ideal of $M(S)$.

Proposition 2.3 *Let S be a commutative semigroup with identity. if $M(S)$ is weakly amenable, then so is $M_0(S)$.*

Proof. Since S is a semigroup with identity, from Proposition 2.1 of [9], it follows that $\overline{M_0(S)^2} = M_0(S)$. Now the proof is complete by Theorem 2.8.69 of [2]. $\square$

Proposition 2.4 *Let S be a commutative foundation semigroup with identity. Let $M_a(S)$ is weakly amenable, then any one co-dimensional ideal I of $M_a(S)$ is weakly amenable.*

Proof. In this case from Proposition 2.3 of [9], it follows that $\overline{I^2} = I$. Now the proof is complete by by Theorem 2.8.69 of [2]. $\square$

Recall that $LUC(S)$ be the space of all function $g \in C_b(S)$, that the mapping $x \mapsto {}_xg$ from S into $C_b(S)$ is $\|.\|_\infty$-continuous, where $C_b(S)$ denotes the space of all bounded continuous complex-valued functions on S, and ${}_xg(y) = g(xy)$ for all $x, y \in S$. The following result gives a generalization of Theorem 2.1 from [4]. We note that $M(S)$ is not weak amenable when S is a locally compact group, in general.

Theorem 2.5 *Let S be a foundation semigroup with identity such that $C^{-1}D$ and CD^{-1} is a compact subset of S for every tow compact subset C and D of S. Then weakly amenable of $M_a(S)^{**}$ implies weak amenability of $M(S)$.*

Proof. First we note that by Lemma 1 of [6] $C_0(S)^\perp$ is a closed ideal of $LUC(S)^*$ with $LUC(S)^* = M(S) \oplus C_0(S)^\perp$, where $C_0(S)$ is the subset of $C_b(S)$ consisting of functions vanishing at infinity. Also, since $M_a(S)$ has a bounded approximate identity with norm one (see [5]), there exists a right identity E

of $M_a(S)^{**}$ with $\|E\| = 1$. Now, from ([8], page 1198) and Lemma 2.1 of [5], it follows that $EM_a(S)^{**}$ is isometrically isomorphic with $LUC(S)^*$ and so

$$M_a(S)^{**} = \mathrm{LUC}(S)^* + (I - E)M_a(S)^{**}$$

where I is the identity map on $M_a(S)^{**}$ and $(I - E)M_a(S)^{**}$ is a closed ideal having trivial product. Now suppose that $M(S)$ is not weakly amenable, and let $D : M(S) \to M(S)^*$ be a non-inner derivation. Define $\Delta : LUC(S)^* \to LUC(S)^{**}$ by

$$\Delta(\mu + h) = T_{D(\mu)},$$

where $T_{D(\mu)} \in LUC(S)^{**}$ is given by

$$\langle T_{D(\mu)}, \nu + h\rangle = \langle D(\mu), \nu\rangle \qquad (\nu \in M(S), h \in C_0(S)^{\perp}).$$

Since $C_0(S)^{\perp}$ is an ideal in $LUC(S)^*$, Δ is a derivation. If there is a $\Phi \in LUC(S)^{**}$ with $\Delta = ad_{\Phi}$, then $\Psi = \Phi_{|_{M(S)}}$ is an element of $M(S)^*$ with $D = ad_{\Psi}$. Now the derivation $\Lambda : M_a(S)^{**} \to M_a(S)^{***}$ given by

$$\Lambda(n) = \Delta(En) \qquad (n \in M_a(S)^{**})$$

is not inner. This is a contradiction. □

By Theorem 1.3 of [4], It follows that $M_a(S)^{**}$ is amenable for any finite semigroup S. in the following we show that the converse is true for certain locally compact semigroups. Before, we note that

$$Z_t(M_a(S)^{**}) = \{H \in M_a(S)^{**} : F \longmapsto H \odot F \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

Theorem 2.6 *Let S be a cancellative foundation $*-$semigroup with identity such that $C^{-1}D$ is a compact subset of S for every tow compact subset C and D of S. Then $M_a(S)^{**}$ is amenable if and only if S is finite.*

Proof. It is a well-known result that $M_a(S)^{**}$ has a bounded approximate identity when $M_a(S)^{**}$ be amenable,. Thus by Lemma 2.1 from [4] it follows that $M_a(S)^{**}$ has an identity E. Clearly that the map $F \longmapsto E \odot F$ on $M_a(S)^{**}$ is weak*-weak* continuous, so $E \in Z_t(M_a(S)^{**})$. Now, by Corollary 3.2 of [7] we have $E \in M_a(S)$, so that S must therefore be discrete. The result is now an immediate consequence of Theorem 1.3 of [4]. □

Corollary 2.7 *Let S be a open subsemigroup with identity of a locally compact group G. Then $M(S)^{**}$ is amenable if and only if S is finite.*

Proof. We know that $M_a(S) = M(S)$ for finite semigroup S, so the "if" part is clear. For the converse, it is not hard see that $M_a(S)$ is complemented in $M(S)$, and so $M_a(S)^{**}$ is complemented in $M(S)^{**}$. Moreover, $M_a(S)^{**}$ is an ideal in $M(S)^{**}$, and so is itself amenable if $M(S)^{**}$ is amenable. Now the proof is complete by Theorem 2.6. □

Acknowledgments. The authors would like to thank the referee of the paper for invaluable comments and wish to thank the Azad university of Mobareke for its support.

References

[1] A. C. Baker, and J. W. Baker, *Algebra of measures on a locally compact semigroup III*, J. London Math. Soc. **4** (1972), 685-695.

[2] H. G. Dales, *Banach algebras and automatic continuity*, Clarendon Press, Oxford, (2000).

[3] H. A. Dzinotyweyi, *The analogue of the group algebra for topological semigroups*, Pitman Research Notes in Mathematics Series, London, 1984.

[4] F. Ghahramani, R. J. Loy, and G. A. Willis, *Amenability and weak amenability of second conjugate Banach algebras*, Proc. Amer. Math. Soc. **124** (1996) 1489-1497.

[5] M. Lashkarizadeh Bami, *On the multipliers of* $(M_a(S), L^\infty(S, M_a(S)))$ *of a foundation semigroup* S, Math. Nachr. **181** (1996) 73-80.

[6] M. Lashkarizadeh Bami, *Ideals of* $M(S)$ *as ideals of* $LUC(S)^*$ *of a compactly cancellative semigroup* S, Math. Japon. **48** (1998) 363-366.

[7] M. Lashkarizadeh Bami, *The topological centers of* $LUC^*(S)$ *and* $M_a^{**}(S)$ *of certain semigroups*, Glasgow Math. J. **42** (2000) 335-343.

[8] A. T. Lau, and A. Ulger, *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc. **348** (1996), 1191-1212.

[9] R. Nasr-Isfahani, *Factorization in some ideals of Lau algebras with application to semigroup algebras*, Bull. Belg. Math. Soc. **7** (2000), 429-433.

[10] G. L. Sleijpen, *Convolution of measure algebras on semigroups*, Ph.D. Thesis (Katholike Universiteit), The Netherlands, (1976)

Received: September 16, 2008