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Characterizing Linear Bounded Operators via Integral

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Abstract. Let S be a locally compact space and let X be a Banach space. Let us consider the function space $C_0(S, X)$ of all continuous functions $f : S \to X$ vanishing at infinity, endowed with the uniform topology. We shall be concerned with integral representations of linear bounded operators $T : C_0(S, X) \to X$. The main result is a complete characterization of those operators which enjoy an integral form with respect to a scalar measure μ on S. Furthermore we show that such operators also have an integral representation with respect to an operator valued measure G on S with values in $\mathcal{L}(X, X)$, the space of bounded operators on X. Finally, relationships between the different measures are established and this allows to characterize the operators under consideration by their representing measures.

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1 Introduction

1.1. Let S be a locally compact space equipped with its Borel field $\mathcal{B}_S$ and let X be a Banach space. A function $f : S \to X$ is said to vanish at infinity, if for every $\varepsilon > 0$ there exists a compact set $K = K_{\varepsilon,f} \subset S$ such that $\|f(x)\| < \varepsilon$, $\forall x \notin K$. We shall denote by $C_0(S, X)$ the vector space of all continuous functions $f : S \to X$ vanishing at infinity. If $X = \mathbb{R}$, we note $C_0(S, X) = C_0(S)$.
For $f \in C_0(S, X)$, we put:

$$(1.2) \qquad \|f\| = \underset{s\in S}{Sup}\, \|f(s)\|$$

Then it is well known that:

1.3. Proposition: *Formula* (1.2) *defines a norm on* $C_0(S,X)$*, for which* $C_0(S,X)$ *is a Banach space.*

For a Banach space E, E^* will be the topological dual of E and E^{**} the topological dual of E^*. It is assumed that all measures considered here are defined on $\mathcal{B}_S$. If μ is a set function on $\mathcal{B}_S$ with values in the Banach space E we refer the reader to Chap.IV of [4] and Chap. I of [1] for the definition of the variation $v(\mu)$ and the semivariation $\|\mu\|$ of μ.

1.4 We shall deal occasionally with additive set functions $G : \mathcal{B}_S \to \mathcal{L}(X,E)$, where $\mathcal{L}(X,E)$ is the space of linear bounded operators of the Banach space X into the Banach space E.
If G is such a function, we define the semivariation of G by the set function:

$$B \in \mathcal{B}_S,\ \widetilde{G}(B) = Sup \left\| \sum_i G(A_i).x_i \right\|$$

the supremum being over all finite partitions $\{A_i\}$ of B in $\mathcal{B}_S$ and all finite systems of vectors $\{x_i\}$ in X, with $\|x_i\| \leq 1\ \forall i$.
The function G is said to be of finite semivariation if $\widetilde{G}(B)$ is finite for all $B \in \mathcal{B}_S$.

The aim of this work is to characterize bounded linear operators $T : C_0(S,X) \to X$ via integral representation either by a scalar measure or by a vector measure. In section 2 we start with the identification of those $T : C_0(S,X) \to X$ that do have an integral form with respect to a scalar measure.This will extend the results of [7] to the setting of a locally compact space S. In section 3 we will consider integral representation by operator valued measures for a simple integration process. Finally relationships between the various representations are considered.

2- The class $\mathcal{R}$

In this section and in section 3 we assume that X is a fixed Banach space and S a locally compact space.
Let μ be a bounded signed measure on S an let us consider the Bochner integral

$$f \in C_0(S,X), \quad T_\mu f = \int_S f\, d\mu \tag{2.1}$$

For all properties of the Bochner integral we refer the reader to [5].

2.2 Proposition: (a) *Formula* (2.1) *defines a linear bounded operator from* $C_0(S,X)$ *into* X*.*
(b) *For every bounded operator* $U : X \to E$ *from* X *into the Banach space* E *we have* $UT_\mu f = T_\mu Uf$ *for all* $f \in C_0(S,X)$*, where* Uf *is the vector of* $C_0(S,E)$ *given by* $Uf\,(t) = U\,(f\,(t))\,, t \in S$*.*
(c) $\|T_\mu\| = v\,(\mu)$ *(the variation of* μ *).*

Now we want to identify those operators $T : C_0(S, X) \to X$, for which there is a signed measure μ on S such that $T = T_\mu$. To this end we shall extend the strategy used in [7] to the present setting. So we begin by introducing the family of bounded operators $U_{x^*} : C_0(S, X) \to C_0(S)$, $x^* \in X^*$ given by:

$$f \in C_0(S, X), \quad U_{x^*} f = x^* \circ f, \tag{2.3}$$

where $x^* \circ f(t) = x^*(f(t))$, $t \in S$. We collect some facts about U_{x^*} for later use:

2.4. Proposition: *U_{x^*} is onto for each $x^* \neq 0$.*

Proof: If $x^* \neq 0$, take $x \in X$ such that $x^*(x) = 1$. For $h \in C_0(S)$ put $f(s) = h(s).x$, $s \in S$. Then $f \in C_0(S, X)$ and we have $U_{x^*} f(s) = x^*(f(s)) = h(s).x^*(x) = h(s)$.■

2.5. Definition: Let $\mathcal{R}$ be the class of linear bounded operators $T : C_0(S, X) \to X$ satisfying the following condition:

$$(C) \qquad x^*, y^* \in X^*,\ f, g \in C_0(S, X) : U_{x^*} f = U_{y^*} g \Longrightarrow x^* T f = y^* T g$$

It is easy to check that $\mathcal{R}$ is a closed subspace of the space of all bounded operators $T : C_0(S, X) \to X$. Note also that every T_μ in (2.1) is in $\mathcal{R}$, by 2.2(b).

The outstanding fact about $\mathcal{R}$ is:

2.6. Theorem: *Let T be an operator in $\mathcal{R}$, then there exists a unique bounded linear functional $\varphi : C_0(S) \to \mathbb{R}$ such that:*

$$\varphi \circ U_{x^*} = x^* \circ T \tag{2.7}$$

for every $x^* \in X^*$.

Proof: Let $h \in C_0(S)$ and $x^* \in X^*$, $x^* \neq 0$; by 2.4 there is an $f \in C_0(S, X)$ such that $U_{x^*} f = h$. then we put:

$$\varphi(h) = x^* T f \tag{2.8}$$

If $U_{x^*} f = U_{y^*} g = h$, then $x^* T f = y^* T g$, by condition (C); so φ is well defined, and it is easy to see that φ is linear. We must show that φ is bounded. We may argue as follows: since U_{x^*} is bounded and onto, by the open mapping principle there exists a constant $K = K_{x^*} > 0$ such that for every $h \in C_0(S)$, there is a solution $f \in C_0(S, X)$ of $U_{x^*} f = h$, with $\|f\| \leq K. \|h\|$. From (2.8) we deduce that $\|\varphi(h)\| \leq \|x^*\| \|T\| \|f\| \leq \|x^*\| \|T\| K \|h\|$, which proves that φ is bounded.

It is noteworthy that the functional φ does not depend on the choice of x^* but depends only on T. For if φ_{x^*} and φ_{y^*} are defined as in (2.8), with $x^*, y^* \neq 0$, then $\varphi_{x^*}(h) = x^* T f$ if $h = U_{x^*} f$ and $\varphi_{y^*}(h) = y^* T g$ if $h = U_{y^*} g$; but condition (C) on T implies that $\varphi_{x^*}(h) = \varphi_{y^*}(h)$. It remains to prove (2.7). For $f \in C_0(S, X)$ and $x^* \in X^*$, we have $h = U_{x^*} f \in C_0(S)$, and (2.8) gives $\varphi(h) = \varphi(U_{x^*} f) = x^* T f$. Since f and x^* are arbitrary, (2.7) follows.■

By a straightforward extension of theorem 2.1 in [7], to the present setting, we get:

2.9. Theorem: *There is an isometric isomorphism between the Banach space* $\mathcal{R}$ *and the topological dual* $C_0^*(S)$ *of* $C_0(S)$*, for each non trivial Banach space* X.

Now we turn to the representation of operators T in the class $\mathcal{R}$,via Bochner integrals.

2.10. Theorem: *Every operator* T *in* $\mathcal{R}$ *is of the form* T_μ*, for a unique bounded signed measure* μ *on* S*. In other words, to each* T *in the class* $\mathcal{R}$ *there corresponds a unique bounded signed measure* μ *on* S *such that:*

$$\forall\ f \in C_0(S,X),\ T(f) = \int_S f\ d\mu \qquad (2.11)$$
$$\|T\| = v(\mu).$$

Proof: Let φ the functional corresponding to the operator T according to 2.6, and let μ the measure related to φ by the Riesz representaion theorem ([10] theorem 6.19). Then for every $h \in C_0(S)$, we have $\varphi(h) = \int_S h\ d\mu$; so, if $h = U_{x^*}f$ with $f \in C_0(S,X)$, we get, by 2.2 (b) : $\varphi(U_{x^*}f) = x^*T_\mu f$. But by 2.7 $\varphi(U_{x^*}f) = x^*Tf$, therefore $x^*Tf = x^*T_\mu f$; and since f and x^* are arbitrary, we deduce that $T = T_\mu$.■

3-Representations by operator valued measures

In this section we show that a linear bounded operator $T : C_0(S,X) \to X$ in the class $\mathcal{R}$ enjoy an integral representation by an operator valued measure whose special form will be used to characterize the operator T as being a member of the class $\mathcal{R}$. In what follows we first make precise the integration process which will be used in our representation.

3.1 A simple measurable function s on S with values in the Banach space X is a function of the form $s(\bullet) = \sum\limits_i 1_{A_i}(\bullet).x_i$, where $\{A_i\}$ is a finite partition of S in $\mathcal{B}_S$, and $\{x_i\}$ is a finite system of vectors in the Banach space X. The symbol 1_{A_i} means the characteristic function of the set A_i. Let $\mathfrak{S}$ be the set of all $X-$valued simple functions on S. A function $f : S \to X$ is said to be measurable if there is a sequence s_n in $\mathfrak{S}$ converging uniformly to f on S. Let $\mathcal{M}(S,X)$ be the set of all measurable functions. Then $\mathfrak{S}$ and $\mathcal{M}(S,X)$ are in an obvious way vector subspaces of the space $\mathcal{F}$of all bounded functions $f : S \to X$. Actually the uniform convergence alluded to above is the convergence with respect to the supremum norm $\|f\| = \underset{s\in S}{Sup}\|f(s)\|$, for $f \in \mathcal{F}$. It is the same to say that $\mathcal{M}(S,X)$ is the closure of $\mathfrak{S}$ for this norm.

Now let $G : \mathcal{B}_S \to \mathcal{L}(X,E)$ be an additive set function on $\mathcal{B}_S$ with values in $\mathcal{L}(X,E)$, the space of linear bounded operators from the Banach space X into the Banach space E. Assume that G has finite semivariation (see 1.4).

We define the integral of the simple function $s(\bullet) = \sum_i 1_{A_i}(\bullet).x_i$ over the set $B \in \mathcal{B}_S$, with respect to G by:

$$\int_B s\ dG = \sum_i G(A_i \cap B).x_i \tag{3.2}$$

It is easy to check that the integral is well defined and satisfies:

$$\left\| \int_B s dG \right\| \leq \|s\| .\widetilde{G}(B) \tag{3.3}$$

Let us observe that estimation (3.3) implies that the linear operator U_B : $\mathfrak{S} \rightarrow E$, with $U_B(s) = \int_B s\ dG$ is bounded. So we can extend it in a unique manner to a bounded operator on the closure $\mathcal{M}(S,X)$ of $\mathfrak{S}$. This extension will be our integration process on the space $\mathcal{M}(S,X)$ of measurable functions.We shall denote it also by U_B with $U_B = U$ if $B = S$. Note that if $f \in \mathcal{M}(S,X)$ and if s_n is a sequence in $\mathfrak{S}$ such that $\|f - s_n\| \rightarrow 0$ then the integral of f is given by:

$$U_B(f) = \int_B f\ dG = \lim_n \int_B s_n\ dG \tag{3.4}$$

By (3.3) the integral (3.4) does not depend on the sequence s_n chosen to converge to the function f. This simple integration process will be sufficient for our purpose. The outstanding facts are summarized in the following:

3.5 Theorem: *Let G be an additive $\mathcal{L}(X,E)$-valued set function with finite semivariation on $\mathcal{B}_S$.Then:*
(a) The integral $\int_B f\ dG$ is linear in $f \in \mathcal{M}(S,X)$ and satisfies:

$$\widetilde{G}(B) = Sup\left\{ \left\| \int_B f\ dG \right\|, \|f\| \leq 1, \quad f \in \mathcal{M}(S,X) \right\} \tag{3.6}$$

in other words the operator $U_B : \mathcal{M}(S,X) \rightarrow E$ given by $U_B(f) = \int_B f\ dG$ is bounded with norm $\|U_B\| = \widetilde{G}(B)$, for each $B \in \mathcal{B}_S$. Conversely:
(b) Let $U : \mathcal{M}(S,X) \rightarrow E$ be a bounded operator. Then there is a unique additive set function $G : \mathcal{B}_S \rightarrow \mathcal{L}(X,E)$,
with finite semivariation such that:

$$\forall\ f \in \mathcal{M}(S,X),\ \ \forall\ B \in \mathcal{B}_S, \quad U(f.1_B) = \int_B f\ dG \tag{3.7}$$

(c) Let $\Lambda : E \rightarrow Y$ be a bounded operator from E into the Banach space Y. Let us define $\Lambda G : \mathcal{B}_S \rightarrow \mathcal{L}(X,Y)$ by $(\Lambda G)(B)x = \Lambda(G(B)x)$, $B \in \mathcal{B}_S$, $x \in X$. Then ΛG is an additive $\mathcal{L}(X,Y)$-valued set function with finite semivariation and we have:

$$\forall\ f \in \mathcal{M}(S,X), \quad \int_S f\ d\Lambda G = \Lambda\left(\int_S f dG\right) \tag{3.8}$$

Proof: (a) To prove (3.6) start with f simple and use (3.2) and the definition of $\widetilde{G}(B)$. For general f use (3.4).
(b) Define $G : \mathcal{B}_S \to \mathcal{L}(X, E)$ by $G(B).x = U(1_B.x)$, for $B \in \mathcal{B}_S$, and $x \in X$. Then G is additive since U is linear and G is $\mathcal{L}(X, E)$-valued because U is bounded. Now (3.7) is easily checked by (3.2) and (3.4).
(c) To prove (3.8) start with f simple and use the definition of ΛG, then apply (3.4), (recall that the operator Λ is bounded).■

Actually, part (b) of this theorem is an integral representation of a bounded operator U on the space $\mathcal{M}(S, X)$ by means of an $\mathcal{L}(X, E)$-valued set function G on $\mathcal{B}_S$. In our context we need representation for operators on the space of continuous functions $C_0(S, X)$. This is a less trivial problem which will be solved presently for operators in the class $\mathcal{R}$. First let us observe:

3.9 Proposition: *We have*

$$C_0(S, X) \subset \mathcal{M}(S, X),$$

that is $C_0(S, X)$ is a subspace of $\mathcal{M}(S, X)$.

Proof: Let us consider the set of all functions $f \in C_0(S, X)$ of the form $f(\bullet) = g(\bullet).x$, with $g \in C_0(S)$ and x fixed in X. We denote by $C_0(S) \otimes X$ the subspace spanned by this set of functions.Then it is known that $C_0(S) \otimes X$ is dense in $C_0(S, X)$ (see Proposition1 §19 in [2]).
So it is enough to show that $C_0(S) \otimes X \subset \mathcal{M}(S, X)$. Let $f \in C_0(S) \otimes X$ of the form $f = g.x$ with $g \in C_0(S)$. Since g is scalar measurable, bounded on S , there exists a sequence t_n of simple scalar functions on S converging to g uniformly on S. But then the functions $s_n = t_n.x$ are simple $X-$valued and converge uniformly to f on S. Thus $f \in \mathcal{M}(S, X)$. ■

Now we are in a position to give the main theorem of this section:

3.10 Theorem: *To each operator $T : C_0(S, X) \to X$ in the class $\mathcal{R}$ there corresponds a unique operator valued set function G on $\mathcal{B}_S$ with values in $\mathcal{L}(X, X)$ such that*

$$\forall f \in C_0(S, X), \qquad Tf = \int_S f \, dG \tag{3.11}$$

Moreover the function G is $\sigma-$additive in the uniform operator topology and takes its values in a compact set of $\mathcal{L}(X, X)$.

Proof: Let μ be the bounded signed measure attached to the operator T by theorem 2.10. Define the set function $G : \mathcal{B}_S \to \mathcal{L}(X, X)$ by:

$$B \in \mathcal{B}_S \,, \qquad G(B) = \mu(B).I. \tag{3.12}$$

where I is the identity operator of X. From (3.12) we deduce that the semi-variation of G is equal to the variation of μ and so it is finite. It is a simple routine job to check that the integration process (3.2) – (3.4) with this G is exactly the Bochner integration process with the measure μ. This means that for every $f \in \mathcal{M}(S,X)$ we have $\int_S f\ dG = \int_S f\ d\mu$. Note that the Bochner integral in the RHS is well defined since the functions f and μ are both bounded. From (2.11) we deduce the validity of (3.11). On the otherhand formula (3.12) shows that G is $\sigma-$additive in the uniform operator topology of $\mathcal{L}(X,X)$ and takes values in the one dimensional subspace of $\mathcal{L}(X,X)$ spanned by I . But the range of G is bounded by the variation $v(\mu)$ of μ, so it is conditionally compact.■

Let us point out the converse:

3.13 Theorem: *Let $T : C_0(S,X) \to X$ be a bounded operator such that* (3.11) *holds with G given by* (3.12), *where μ is a bounded scalar measure. Then $T \in \mathcal{R}$, that is $T = T_\mu$.*

Proof: For $f \in C_0(S,X)$, we have $Tf = \int_S f\ dG$ and, by (3.8) with $\Lambda = x^*$, we have $x^*Tf = \int_S f\ dx^*G$, for each $x^* \in X^*$. From (3.12) we deduce that $x^*G(\bullet) = \mu(\bullet).x^*$, and then $x^*Tf = \int_S f\ d\mu.x^* = \int_S x^* \circ f\ d\mu$, by standard tools of integration. The last integral is equal to $x^*T_\mu f$ by proposition $(2.2),(b)$. Thus $x^*Tf = x^*T_\mu f$; since f and x^* are arbitrary, we deduce that $T = T_\mu$.■

For the sake of completeness, we give the following integral representation valid for operators $T \in \mathcal{R}$ in the case S compact. For the proof the reader is refered to theorem 5.3 in[7]:

3.14 Theorem:*Suppose S compact. Then a linear bounded operator $T : C_0(S,X) \to X$ is in the class $\mathcal{R}$ if and only if T admits an integral representation by the set function $G : \mathcal{B}_S \to \mathcal{L}(X,X^{**})$, such that:*

$$B \in \mathcal{B}_S\ , \qquad G(B) = \mu(B).\gamma \tag{3.15}$$

where μ is a bounded scalar measure and $\gamma : X \to X^{**}$ the canonical isomorphism.

Proof: The necessity comes from theorem 5.3 of [7]. The proof of the sufficiency is a straightforward adaptation of the proof of theorem 3.13.■

3.16 Theorem: *If S is compact, then for each $T \in \mathcal{R}$, the representing measure G in* (3.15) *is $\sigma-$additive in the uniform topology of $\mathcal{L}(X,X^{**})$ and takes its values in a compact set of $\mathcal{L}(X,X^{**})$.*

Proof: We have: $G(\bullet) = \mu(\bullet).\gamma$, where μ is a bounded scalar measure and γ is the canonical isomorphism of X into X^{**}.

It is clear, that G is $\sigma-$additive in the uniform topology of $\mathcal{L}(X,X^{**})$. On the other hand, the values $G(A) = \mu(A).\gamma$, $A \in \mathcal{B}_S$, of the measure G, form a subset B of the space K_γ spanned by γ in $\mathcal{L}(X,X^{**})$. Since K_γ is one dimensional and since B is bounded by $v(\mu)$, we deduce that B is conditionally compact in K_γ and also in $\mathcal{L}(X,X^{**})$.■

Another question one can ask about operators in the class $\mathcal{R}$ concerns weak compactness properties. Unfortunetely, we have been unable to prove that such operators are weakly compact, even though we strongly feel they are so. However we have the following partial result.

3.17 Theorem: *Let T be an operator in the class $\mathcal{R}$. Then for each $x \in X$, the operator $T_x : C_0(S) \to X$ defined by: $T_x(g) = T(g.x)$, $g \in C_0(S)$, is weakly compact.*

Proof: Since $T \in \mathcal{R}$ we have by (2.11) $T(g.x) = \int_S g.x \; d\mu$. So we deduce $T_x(g) = \int_S g. \; d\mu(\bullet).x$, where $\mu(\bullet).x$ is a vector measure (in the sense of IV-10 in [4]). It is enough to show weakcompactness for the adjoint operator $T_x^* : X^* \to C_0^*(S)$. From theorem IV-10-8 (f) in [5]), we have $x^* T_x g = \int_S g dx^* (\mu(\bullet).x)$ for all $x^* \in X^*$. So we can write $T_x^*(x^*) = x^*(\mu(\bullet).x)$. Since the set of numerical measures $\{x^*(\mu(\bullet).x) : x^* \in X^*, \quad \|x^*\| \leq 1\}$ is weakly compact (by IV-10-2 [4]) we deduce that T_x^* is weakly compact.■

For integral representations in a more general setting see [3], [8], [9].

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