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Generalized Rough Sets and Relations

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Abstract

In this paper, we introduce the notions of generalized upper (lower) approximations, generalized upper (lower) bounded operations and various operations. We investigate their properties.

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1 Introduction and preliminaries

Mathematics requires that all mathematical notions (including set) must be exact, otherwise precise reasoning would be impossible. However, philosophers and recently computer scientists as well as other researcher have become interested in vague concepts. One of them is a view of rough set. Rough set theory was introduced by Pawlak [2-6] to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation. Yao [7,8] investigated the connections among binary relations, neighborhood operators and rough approximation operators.

In this paper, we introduce the notions of generalized upper (lower) approximations, generalized upper (lower) bounded operations and various operations. We investigate their properties.

2 Preliminaries

Definition 2.1 A subset R of $X \times X$ is an equivalence relation on X if it satisfies the following conditions:

(reflexive) $(x, x) \in R$ for all $x \in X$,

(symmetric) If $(x, y) \in R$, then $(y, x) \in R$,

(transitive) If $(x, y), (y, z) \in R$, then $(x, z) \in R$.

A subset R of $X \times X$ is a quasi-equivalence relation if it is reflexive and transitive.

Let $R_1 \subset P(U \times V)$ and $R_2 \subset P(V \times W)$ be relations. The compositions of R_1 and R_2 are defined as

$$R_1 \circ R_2 = \{(x, z) \in U \times W \mid (\exists y \in V)((x, y) \in R_1,\ (y, z) \in R_2)\}$$

$$R_1 \Rightarrow R_2 = \{(x, z) \in U \times W \mid (\forall y \in V)((x, y) \in R_1 \rightarrow (y, z) \in R_2)\}$$

$$R_1 \Leftarrow R_2 = \{(x, z) \in U \times W \mid (\forall y \in V)((y, z) \in R_2 \rightarrow (x, y) \in R_1)\}$$

$$R_1 \oplus R_2 = \{(x, z) \in U \times W \mid (\forall y \in V)((x, y) \in R_1 \vee (y, z) \in R_2)\}$$

$$R_1^s = \{(y, x) \in V \times U \mid (x, y) \in R_1\}.$$

Example 2.2 Let $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ and $W = \{z_1, z_2\}$. Let $R_1 = \{(x_1, y_1), (x_2, y_1)\}$ and $R_2 = \{(y_1, z_2), (y_2, z_2)\}$. Then we obtain:

$$R_1 \circ R_2 = \{(x_1, z_2), (x_2, z_2)\},\ R_1 \Rightarrow R_2 = \{(x_1, z_2), (x_2, z_2)\}$$

$$R_1 \Leftarrow R_2 = \{(x_1, z_1), (x_2, z_1)\},\ \ R_1 \oplus R_2 = \{(x_1, z_2), (x_2, z_2)\}.$$

Lemma 2.3 *Let $R \subset P(U \times U)$ be a relation. We have the following properties.*

(1) *If R is reflexive, then $R \subset R \circ R$, $(R \Rightarrow R) \subset R$, $(R \Leftarrow R) \subset R$ and $R \circ R$ is reflexive.*

(2) *$(R \circ R)^c = R^c \oplus R^c$. If R^c is reflexive, then $R \oplus R \subset R$.*

(3) *R is transitive iff $R \circ R \subset R$. Moreover, R^c is transitive iff $R \subset R \oplus R$.*

(4) *If $R^s \circ R \subset R$, then $R \subset (R \Rightarrow R)$. Moreover, if $R \circ R^s \subset R$, then $R \subset (R \Leftarrow R)$.*

(5) *If R is a quasi-equivalence relation, then $R = (R^s \Rightarrow R)$, $R = (R \Leftarrow R^s)$, $R = R \circ R$ and $R^c = R^c \oplus R^c$.*

(6) *$R^s \circ R$ and $R^s \oplus R$ are symmetric.*

(7) *$R^s \circ R \subset R$ and R is reflexive iff R is an equivalence relation.*

(8) *$R \subset R^s \oplus R$ and R^c is reflexive iff R^c is an equivalence relation.*

(9) *If R is an equivalence relation, then $R = R \circ R = R \Rightarrow R = R \Leftarrow R = R^s \circ R = R^s \Rightarrow R = R \Leftarrow R^s$ and $R^c = R^c \oplus R^c = (R^c)^s \oplus R^c$.*

Proof. (1) Let $(x,z) \in (R \Rightarrow R)$. Then $(x,x) \in R$ implies $(x,z) \in R$. Thus, $(R \Rightarrow R) \subset R$.

(2) Since R^c is reflexive, by (1), $R^c \subset R^c \circ R^c$. Thus $R \oplus R = (R^c \circ R^c)^c \leq R$.

(5) Let $(x,z) \in (R^s \Rightarrow R)$. Since R is reflexive, $(x,x) \in R^s$ implies $(x,z) \in R$. Thus, $(R^s \Rightarrow R) \subset R$. Since R is transitive;i.e, $(x,z) \in R \wedge (z,y) \in R$ implies $(x,y) \in R$, equivalently, $(z,y) \in R$ implies $[(x,z) \in R \rightarrow (x,y) \in R]$ for all $x \in U$. Thus $(z,y) \in (R^s \Rightarrow R)$ Therefore $(R^s \Rightarrow R) = R$.

(6) It follows from

$$\begin{aligned}(x,z) \in R^s \circ R \quad &\text{iff } (\exists y \in U)((x,y) \in R^s \wedge (y,z) \in R)\\ &\text{iff } (\exists y \in U)((z,y) \in R^s \wedge (y,x) \in R)\\ &\text{iff } (z,x) \in R^s \circ R.\end{aligned}$$

Other cases are easily proved.

3 Generalized rough sets and relations

Definition 3.1 Let U and V be sets with a relation $R \in P(U \times V)$. For each $A \subset U$ and $B \subset V$, we define:

(1) A lower approximation $\underline{apr}_{R_u} : P(V) \rightarrow P(U)$ is defined as:

$$\underline{apr}_{R_u}(B) = \{x \in U \mid (\forall y \in V)\ ((x,y) \in R \rightarrow y \in B)\}$$

and an upper approximation $\overline{apr}_{R_u} : P(V) \rightarrow P(U)$ is defined as:

$$\overline{apr}_{R_u}(B) = \{x \in U \mid (\exists y \in V)\ ((x,y) \in R \wedge y \in B)\}.$$

The pair $(\underline{apr}_{R_u}, \overline{apr}_{R_u})$ is called a generalized rough approximation on U.

(2) A lower approximation $\underline{apr}_{R_v} : P(U) \rightarrow P(V)$ is defined as:

$$\underline{apr}_{R_v}(A) = \{y \in V \mid (\forall x \in U)\ ((x,y) \in R \rightarrow x \in A)\}$$

and an upper approximation $\overline{apr}_{R_v} : P(U) \rightarrow P(V)$ is defined as:

$$\overline{apr}_{R_v}(A) = \{y \in V \mid (\exists x \in U)((x,y) \in R \wedge x \in A)\}.$$

The pair $(\underline{apr}_{R_v}, \overline{apr}_{R_v})$ is called a generalized rough approximation on V.

(3) A map $Ib_{R_u} : P(V) \rightarrow P(U)$ is called a generalized lower bound operator defined as:

$$Ib_R(B) = \{x \in U \mid (\forall y \in V)(y \in B \rightarrow (x,y) \in R)\}$$

and a map $Ub_{R_v} : P(U) \to P(V)$ is called a generalized upper bound operator defined as

$$Ub_R(A) = \{y \in V \mid (\forall x \in U)(x \in A \to (x,y) \in R)\}.$$

(4) A map $O_{R_u} : P(V) \to P(U)$ is defined as:

$$O_{R_u}(B) = \{x \in U \mid (\forall y \in V)(y \in B \vee (x,y) \in R)\}$$

and a map $O_{R_v} : P(U) \to P(V)$ is defined as:

$$O_{R_v}(A) = \{y \in V \mid (\forall x \in U)(x \in A \vee (x,y) \in R\}.$$

(5) A map $P_{R_u} : P(V) \to P(U)$ is defined as

$$P_{R_u}(B) = \{x \in U \mid (\exists y \in V)(y \in B \wedge (x,y) \notin R)\}$$

and a map $P_{R_v} : P(U) \to P(V)$ is defined as

$$P_{R_u}(A) = \{y \in V \mid (\exists x \in U)(x \in A \wedge (x,y) \notin R)\}.$$

Theorem 3.2 *Let U and V be sets with a relation $R \in P(U \times V)$. Then the following properties hold:*

(1) $\underline{apr}_{R_u}(B) = O_{R_u^c}(B) = Ib_{R^c}(B^c)$, *for all* $B \in P(V)$.
(2) $\underline{apr}_{R_v}(A) = O_{R_v^c}(A) = Ub_{R^c}(A^c)$, *for all* $A \in P(U)$.
(3) $(\overline{apr}_{R_u}(B))^c = Ib_{R^c}(B) = O_{R_u^c}(B^c) = \underline{apr}_{R_u}(B^c)$, *for all* $B \in P(V)$.
(4) $(\overline{apr}_{R_v}(A))^c = Ub_{R^c}(A) = O_{R_v}(A^c) = \underline{apr}_{R_v}(A^c)$, *for all* $A \in P(U)$.
(5) $O_{R_u}(B) = \underline{apr}_{R_u^c}(B) = Ib(B^c)$, *for all* $B \in P(V)$.
(6) $O_{R_v}(A) = \underline{apr}_{R_v^c}(A) = Ub(A^c)$, *for all* $A \in P(U)$.
(7) $P_{R_u}(B) = (\underline{apr}_{R_u}(B^c))^c = \overline{apr}_{R_u^c}(B)$, *for all* $B \in P(V)$.
(8) $P_{R_v}(A) = \overline{apr}_{R_v^c}(A) = (\underline{apr}_{R_v}(A^c))^c$, *for all* $A \in P(U)$.

Proof. (1)

$$\begin{aligned}
\underline{apr}_{R_u}(B) &= \{x \in U \mid (\forall y \in V)\ ((x,y) \in R \to y \in B)\} \\
&= \{x \in U \mid (\forall y \in V)\ ((x,y) \notin R \vee y \in B)\} \\
&= O_{R_u^c}(B) \\
&= \{x \in U \mid (\forall y \in V)\ (y \in B^c \to (x,y) \in R^c)\} \\
&= Ib_{R^c}(B^c)
\end{aligned}$$

(3)

$$\begin{aligned}
(\overline{apr}_{R_u}(B))^c &= (\{x \in U \mid (\exists y \in V)\ ((x,y) \in R \wedge y \in B)\})^c \\
&= \{x \in U \mid (\forall y \in V)\ ((x,y) \notin R \vee y \notin B)\} \\
&= O_{R_u^c}(B^c) \\
&= \{x \in U \mid (\forall y \in V)\ (y \in B \to (x,y) \in R^c)\} \\
&= Ib_{R^c}(B) \\
&= \{x \in U \mid (\forall y \in V)\ ((x,y) \in R \to y \in B^c)\} \\
&= \underline{apr}_{R_u}(B^c)
\end{aligned}$$

(5)

$$\begin{aligned} O_{R_u}(B) &= \{x \in U \mid (\forall y \in V)(y \in B \vee (x,y) \in R)\} \\ &= \{x \in U \mid (\forall y \in V)(y \in B^c \rightarrow (x,y) \in R)\} \\ &= Ib_R(B^c) \\ &= \{x \in U \mid (\forall y \in V)((x,y) \in R^c \rightarrow y \in B)\} \\ &= \underline{apr}_{R_u^c}(B). \end{aligned}$$

Other cases are similarly proved.

Example 3.3 Let $U = \{x_1, x_2, x_3\}$, $V = \{y_1, y_2, y_3\}$, $A = \{x_2, x_3\}$ and $B = \{y_2, y_3\}$ be sets. Let $R = \{(x_1, y_1), (x_1, y_3), (x_2, y_1), (x_3, y_2)\}$ be a relation. Then we obtain:

$$\underline{apr}_{R_u}(B) = \{x_3\},\ \underline{apr}_{R_u^c}(B) = \{x_1, x_2\},$$

$$\overline{apr}_{R_u}(B) = \{x_1, x_3\},\ \overline{apr}_{R_u^c}(B) = \{x_1, x_2, x_3\},$$

$$Ib_R(B) = \underline{apr}_{R_u^c}(B^c) = \emptyset,$$

$$O_{R_u}(B) = \{x_1, x_3\}, P_{R_u}(B) = U,$$

$$\underline{apr}_{R_v}(A) = \{y_2\},\ \underline{apr}_{R_v^c}(A) = \{y_1, y_3\},$$

$$\overline{apr}_{R_v}(A) = \{y_1, y_2\},\ \overline{apr}_{R_v^c}(A) = \{y_1, y_2, y_3\},$$

$$Ub_R(A) = \underline{apr}_{R_v^c}(A^c) = \emptyset,$$

$$O_{R_v}(A) = \{y_1, y_3\}, P_{R_v}(A) = V.$$

Theorem 3.4 *Let $R_1 \subset P(U \times V)$ and $R_2 \subset P(V \times W)$ be two relations. We have the following properties.*

(1) $\underline{apr}_{(R_1 \circ R_2)_u} = \underline{apr}_{R_{1u}} \circ \underline{apr}_{R_{2v}}$ *and* $\underline{apr}_{(R_1 \circ R_2)_w} = \underline{apr}_{R_{2w}} \circ \underline{apr}_{R_{1v}}$.

(2) $\overline{apr}_{(R_1 \circ R_2)_u} = \overline{apr}_{R_{1u}} \circ \overline{apr}_{R_{2v}}$ *and* $\overline{apr}_{(R_1 \circ R_2)_w} = \overline{apr}_{R_{2w}} \circ \overline{apr}_{R_{1v}}$.

(3) $Ib_{(R_1 \Rightarrow R_2)} = \underline{apr}_{R_{1u}} \circ Ib_{R_2}$.

(4) $Ub_{(R_1 \Rightarrow R_2)} = Ub_{R_2} \circ \overline{apr}_{R_{1v}}$.

(5) $Ib_{(R_1 \Leftarrow R_2)} = Ib_{R_1} \circ \overline{apr}_{R_{2v}}$.

(6) $Ub_{(R_1 \Leftarrow R_2)} = \underline{apr}_{R_{2w}} \circ Ub_{R_1}$.

(7) $O_{(R_1 \Rightarrow R_2)_u} = \underline{apr}_{R_{2w}^c} \circ O_{R_{1v}^c} = O_{R_{2w}} \circ O_{R_{1v}^c}$ *and* $O_{(R_1 \Rightarrow R_2)_w} = \underline{apr}_{R_{1u}} \circ \underline{apr}_{R_{2v}^c}$.

(8) $Ib_{R_1 \oplus R_2} = \underline{apr}_{R_{1u}^c} \circ Ib_{R_2} = O_{R_{1u}} \circ Ib_{R_2} = Ib_{R_1} \circ Ib_{R_2}^c$.

(9) $Ub_{R_1 \oplus R_2} = \underline{apr}_{R_{2w}^c} \circ Ub_{R_1} = O_{R_{2w}} \circ Ub_{R_1} = Ub_{R_2} \circ Ub_{R_1}^c$.

(10) $O_{(R_1 \oplus R_2)_u} = O_{R_{1u}} \circ O_{R_{2v}}$ *and* $O_{(R_1 \oplus R_2)_w} = O_{R_{2w}} \circ O_{R_{1v}}$.

(11) $P_{(R_1 \oplus R_2)_u} = P_{R_{1u}} \circ P_{R_{2v}}$ *and* $P_{(R_1 \oplus R_2)_w} = P_{R_{2w}} \circ P_{R_{1v}}$.

(12) $Ib_{R_1} \circ Ib_{R_2} = \underline{apr}_{R_{1u}^c} \circ \overline{apr}_{R_{2u}^c}$.

(13) $Ub_{R_2} \circ Ub_{R_1} = \underline{apr}_{R_{2v}^c} \circ \overline{apr}_{R_{1v}^c}$.

Proof. (1) For each $C \in P(W)$,

$$\begin{aligned}
&\underline{apr}_{(R_1\circ R_2)_u}(C)\\
&= \{x \in U \mid (\forall z \in W)\Big((x,z) \in (R_1 \circ R_2) \to z \in C\Big)\}\\
&= \{x \in U \mid (\forall z \in W)\Big((\exists y \in V)((x,y) \in R_1 \wedge (y,z) \in R_2) \to z \in C\Big)\}\\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\Big(((x,y) \in R_1 \wedge (y,z) \in R_2) \to z \in C\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((x,y) \in R_1 \to (\forall z \in W)((y,z) \in R_2 \to z \in C)\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((x,y) \in R_1 \to y \in \underline{apr}_{R_{2v}}(C))\Big)\}\\
&= \underline{apr}_{R_{1u}}(\underline{apr}_{R_{2v}}(C)).
\end{aligned}$$

(3) For each $C \in P(W)$,

$$\begin{aligned}
&Ib_{(R_1\Rightarrow R_2)}(C)\\
&= \{x \in U \mid (\forall z \in W)\Big(z \in C \to (x,z) \in (R_1 \Rightarrow R_2)\Big)\}\\
&= \{x \in U \mid (\forall z \in W)\Big(z \in C \to (\forall y \in V)((x,y) \in R_1 \Rightarrow (y,z) \in R_2)\Big)\}\\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\Big((z \in C \wedge (x,y) \in R_1) \Rightarrow (y,z) \in R_2\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((x,y) \in R_1 \to (\forall z \in W)(z \in C \Rightarrow (y,z) \in R_2\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((x,y) \in R_1 \to y \in Ib_{R_2}(C)\Big)\}\\
&= \underline{apr}_{R_{1u}}(Ib_{R_2}(C).
\end{aligned}$$

(4)For each $A \in P(U)$,

$$\begin{aligned}
&Ub_{(R_1\Rightarrow R_2)}(A)\\
&= \{z \in W \mid (\forall x \in U)\Big(x \in A \to (x,z) \in (R_1 \Rightarrow R_2)\Big)\}\\
&= \{z \in W \mid (\forall x \in U)\Big(x \in A \to (\forall y \in V)((x,y) \in R_1 \Rightarrow (y,z) \in R_2)\Big)\}\\
&= \{z \in W \mid (\forall x \in U)(\forall y \in V)\Big((x \in A \wedge (x,y) \in R_1) \Rightarrow (y,z) \in R_2)\Big)\}\\
&= \{z \in W \mid (\forall y \in V)\Big((\exists x \in U)(x \in A \wedge (x,y) \in R_1) \Rightarrow (y,z) \in R_2)\Big)\}\\
&= \{z \in W \mid (\forall y \in V)\Big(y \in \overline{apr}_{R_{1v}}(A) \Rightarrow (y,z) \in R_2)\Big)\}\\
&= Ub_{R_2}(\overline{apr}_{R_{1v}}(A))
\end{aligned}$$

(5) For each $C \in P(W)$,

$$\begin{aligned}
&Ib_{(R_1\Leftarrow R_2)}(C)\\
&= \{x \in U \mid (\forall z \in W)\Big(z \in C \to (x,z) \in (R_1 \Leftarrow R_2)\Big)\}\\
&= \{x \in U \mid (\forall z \in W)\Big(z \in C \to (\forall y \in V)((y,z) \in R_2 \to (x,y) \in R_1)\Big)\}\\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\Big(z \in C \to ((y,z) \in R_2 \to (x,y) \in R_1)\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((\exists z \in W)(z \in C \wedge (y,z) \in R_2) \to (x,y) \in R_1)\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big(y \in \overline{apr}_{R_{2v}}(C) \to (x,y) \in R_1)\Big)\}\\
&= Ib_{R_1}(\overline{apr}_{R_{2v}}(C))
\end{aligned}$$

(6) For each $A \in P(U)$,

$$
\begin{aligned}
& Ub_{(R_1 \Leftarrow R_2)}(A) \\
&= \{z \in W \mid (\forall x \in U)\big(x \in A \to (x,z) \in (R_1 \Leftarrow R_2)\big)\} \\
&= \{z \in W \mid (\forall x \in W)\big(x \in A \to (\forall y \in V)((y,z) \in R_2 \to (x,y) \in R_1)\big)\} \\
&= \{z \in W \mid (\forall x \in W)(\forall y \in V)\big(x \in A \to ((y,z) \in R_2 \to (x,y) \in R_1)\big)\} \\
&= \{z \in W \mid (\forall x \in W)(\forall y \in V)\big((y,z) \in R_2 \to (x \in A \to (x,y) \in R_1)\big)\} \\
&= \{z \in W \mid (\forall y \in V)\big((y,z) \in R_2 \to (\forall x \in W)(x \in A \to (x,y) \in R_1)\big)\} \\
&= \{z \in W \mid (\forall y \in V)\big((y,z) \in R_2 \to y \in Ub_{R_1}\big)\} \\
&= \underline{apr}_{R_{2w}}(Ub_{R_1}(A)).
\end{aligned}
$$

(8) For each $C \in P(W)$,

$$
\begin{aligned}
& Ib_{R_1 \oplus R_2}(C) \\
&= \{x \in U \mid (\forall z \in W)\big(z \in C \to (x,z) \in (R_1 \oplus R_2)\big)\} \\
&= \{x \in U \mid (\forall z \in W)\big(z \in C \to (\forall y \in V)((x,y) \in R_1 \vee (y,z) \in R_2)\big)\} \\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\big(z \in C \to ((x,y) \in R_1 \vee (y,z) \in R_2)\big)\} \\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\big(((x,y) \in R_1 \vee (y,z) \in R_2)^c \to z \notin C)\big)\} \\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\big(((x,y) \notin R_1 \wedge (y,z) \notin R_2) \to z \notin C)\big)\} \\
&= \{x \in U \mid (\forall z \in W)(\forall y \in V)\big(((x,y) \notin R_1 \to ((y,z) \notin R_2 \to z \notin C)\big)\} \\
&= \{x \in U \mid (\forall y \in V)\big(((x,y) \notin R_1 \to (\forall z \in W)(z \in C \to (y,z) \in R_2)\big)\} \\
&= \underline{apr}_{R^c_{1u}}(Ib_{R_2}(C)) = O_{R_{1u}}(Ib_{R_2}(C)) \\
&= Ib_{R_1}(Ib^c_{R_2}(C)).
\end{aligned}
$$

(10) For each $C \in P(W)$,

$$
\begin{aligned}
& O_{R_{1u}}(O_{R_{2v}}(C)) \\
&= \{x \in U \mid (\forall y \in V)\big(y \in O_{R_{2v}}(C) \vee (x,y) \in R_1\big)\} \\
&= \{x \in U \mid (\forall y \in V)\big((\forall z \in W)(z \in C \vee (y,z) \in R_2) \vee (x,y) \in R_1\big)\} \\
&= \{x \in U \mid (\forall y \in V)(\forall z \in W)\big((z \in C \vee (y,z) \in R_2) \vee (x,y) \in R_1\big)\} \\
&= \{x \in U \mid (\forall y \in V)(\forall z \in W)\big((z \in C) \vee (\forall y \in V)((y,z) \in R_2) \vee (x,y) \in R_1)\big)\} \\
&= O_{(R_1 \oplus R_2)_u}(C).
\end{aligned}
$$

(11) For each $A \in P(U)$,

$$
\begin{aligned}
&P_{R_{2w}}(P_{R_{1v}}(A)\\
&= \{z \in W \mid (\exists y \in V)\Big(y \in P_{R_{1v}}(A) \wedge (y,z) \notin R_2\Big)\}\\
&= \{z \in W \mid (\exists y \in V)\Big((\exists x \in U)(x \in A \wedge (x,y) \notin R_1) \wedge (y,z) \notin R_2\Big)\}\\
&= \{z \in W \mid (\exists y \in V)(\exists x \in U)\Big((\exists x \in U)(x \in A \wedge ((x,y) \notin R_1 \wedge (y,z) \notin R_2)\Big)\}\\
&= \{z \in W \mid (\exists x \in U)\Big(x \in A \wedge (\exists y \in V)((x,y) \notin R_1 \wedge (y,z) \notin R_2)\Big)\}\\
&= \{z \in W \mid (\exists x \in U)\Big(x \in A \wedge (x,y) \notin (R_1 \oplus R_2)\Big)\}\\
&= Ub_{(R_1 \oplus R_2)_v}(A).
\end{aligned}
$$

(12) For each $C \in P(W)$,

$$
\begin{aligned}
&Ib_{R_1}(Ib_{R_2})(C)\\
&= \{x \in U \mid (\forall y \in V)\Big(y \in Ib_{R_2}(C) \to (x,y) \in R_1\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((\forall z \in W)(z \in C \to (y,z) \in R_2) \to (x,y) \in R_1\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((\forall z \in W)(z \in C \to (y,z) \in R_2) \to (x,y) \in R_1\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((x,y) \notin R_1 \to (\exists z \in W)(z \in C \to (y,z) \in R_2)^c\Big)\}\\
&= \{x \in U \mid (\forall y \in V)\Big((x,y) \notin R_1 \to (\exists z \in W)(z \in C \wedge (y,z) \notin R_2)\Big)\}\\
&= \underline{apr}_{R^c_{1u}}(\overline{apr}_{R^c_{2v}}(C)).
\end{aligned}
$$

Other cases are similarly proved.

Corollary 3.5 *For $R \in P(U \times U)$, we have the following properties.*

(1) $F_{R_u} \circ F_{R_u} = F_{(R \circ R)_u}$ *and* $F_{R_u} \circ F_{R_u} = F_{(R \circ R)_u}$, *for each* $F \in \{\underline{apr}, \overline{apr}\}$.

(2) $F_{R_u} \circ F_{R_v} = F_{(R \circ R^s)_u} = F_{(R \circ R^s)_v}$ *and* $F_{R_v} \circ F_{R_u} = F_{(R^s \circ R)_u} = F_{(R^s \circ R)_v}$, *for each* $F \in \{\underline{apr}, \overline{apr}\}$.

(3) $F_{R_u} \circ F_{R_u} = F_{(R \oplus R)_u}$ *and* $F_{R_u} \circ F_{R_u} = F_{(R \oplus R)_u}$, *for each* $F \in \{O, P\}$.

(4) $F_{R_u} \circ F_{R_v} = F_{(R \oplus R^s)_u} = F_{(R \oplus R^s)_v}$ *and* $F_{R_v} \circ F_{R_u} = F_{(R^s \oplus R)_u} = F_{(R^s \oplus R)_v}$, *for each* $F \in \{O, P\}$.

(5) $Ib_R \circ Ib_R = \underline{apr}_{R^c_u} \circ \overline{apr}_{R^c_u}$.

(6) $Ub_R \circ Ub_R = \underline{apr}_{R^c_v} \circ \overline{apr}_{R^c_v}$.

(7) $Ib_{(R^s \Rightarrow R)} = \underline{apr}_{R_v} \circ Ib_R$.

(8) $Ub_{(R^s \Rightarrow R)} = Ub_R \circ \overline{apr}_{R_u}$.

(9) $Ib_{(R \Leftarrow R^s)} = Ib_R \circ \overline{apr}_{R_v}$.

(10) $Ub_{(R \Leftarrow R^s)} = \underline{apr}_{R_u} \circ Ub_R$.

(11) $O_{(R^s \Rightarrow R)_u} = \underline{apr}_{R^c_v} \circ O_{R^c_u} = O_{R_v} \circ O_{R^c_u}$ *and* $O_{(R^s \Rightarrow R)_v} = \underline{apr}_{R_v} \circ \underline{apr}_{R^c_u}$.

(12) $Ib_{R \oplus R} = \underline{apr}_{R^c_u} \circ Ib_R = O_{R_u} \circ Ib_R = Ib_R \circ Ib^c_R$.

(13) $Ub_{R \oplus R} = \underline{apr}_{R^c_v} \circ Ub_R = O_{R_v} \circ Ub_R = Ub_R \circ Ub^c_R$.

Proof. (2) Since $R \circ R^s$ is symmetric from Lemma 2.3, $\overline{apr}_{(R\circ R^s)_u} = \overline{apr}_{(R\circ R^s)_u}$. Moreover,

$$\begin{aligned}
&\overline{apr}_{R_u}(\overline{apr}_{R_v}(A)) \\
&= \{x \in U \mid (\exists y \in U)\Big(y \in \overline{apr}_{R_v}(A) \wedge (x,y) \in R\Big)\} \\
&= \{x \in U \mid (\exists y \in U)\Big((\exists z \in U)(z \in A \wedge (z,y) \in R) \wedge (x,y) \in R\Big)\} \\
&= \{x \in U \mid (\exists z \in U)\Big(z \in A \wedge (\exists y \in U)((z,y) \in R \wedge (x,y) \in R)\Big)\} \\
&= \{x \in U \mid (\exists z \in U)\Big(z \in A \wedge (\exists y \in U)((z,y) \in R \wedge (y,x) \in R^s)\Big)\} \\
&= \{x \in U \mid (\exists z \in U)\Big(z \in A \wedge (z,x) \in (R \circ R^s)\Big)\} \\
&= \overline{apr}_{(R\circ R^s)_v}(A) = \overline{apr}_{(R\circ R^s)_v}(A).
\end{aligned}$$

Other cases are easily proved from Theorem 3.4.

From Corollary 3.5 and Lemma 2.3, we can obtain the following corollaries.

Corollary 3.6 *Let $R \in L^{U\times U}$ be a quasi-equivalence relation. Then we have the following properties.*

(1) $F_{R_u} \circ F_{R_u} = F_{R_u}$ *and* $F_{R_v} \circ F_{R_v} = F_{R_v}$, *for each* $F \in \{\underline{apr}, \overline{apr}\}$.

(2) $Ib_R = \underline{apr}_{R_v} \circ Ib_R$.

(3) $Ub_R = Ub_R \circ \overline{apr}_{R_u}$.

(4) $Ib_R = Ib_R \circ \overline{apr}_{R_v}$.

(5) $Ub_R = \underline{apr}_{R_u} \circ Ub_R$.

(6) $O_{R_u} = \underline{apr}_{R^c_v} \circ O_{R^c_u} = O_{R_v} \circ O_{R^c_u}$ *and* $O_{R_v} = \underline{apr}_{R_v} \circ \underline{apr}_{R^c_u}$.

Moreover, if R is symmetric, then $F_{R_u} \circ F_{R_v} = F_{R_u} = F_{R_v} = F_{R_v} \circ F_{R_u}$, *for each* $F \in \{\underline{apr}, \overline{apr}\}$.

Corollary 3.7 *Let $R^c \in L^{U\times U}$ be a quasi-equivalence relation. Then we have the following properties.*

(1) $F_{R_u} \circ F_{R_u} = F_{R_u}$ *and* $F_{R_u} \circ F_{R_u} = F_{R_u}$, *for each* $F \in \{O, P\}$.

(2) $Ib_R = \underline{apr}_{R^c_u} \circ Ib_R = O_{R_u} \circ Ib_R = Ib_R \circ Ib^c_R$.

(3) $Ub_R = \underline{apr}_{R^c_v} \circ Ub_R = O_{R_v} \circ Ub_R = Ub_R \circ Ub^c_R$.

Moreover, if R is symmetric, then $F_{R_u} \circ F_{R_v} = F_{R_u} = F_{R_v} = F_{R_v} \circ F_{R_u}$, *for each* $F \in \{O, P\}$.

Example 3.8 Let $M = \{m \in N \mid 1 \leq m \leq 10\}$, $R = \{(m,n) \mid m \leq n, m, n \in M\}$, $A = \{3,5,7\}$ and $B = \{1,2,5,8,9,10\}$ be given. Then R is a quasi-equivalence relation. We obtain:

$$I_R(A) = \underline{apr}_{R^c_v}(A^c) = O_R(B^c) = \{1,2,3\},$$

$$\underline{apr}_{R_u}(A) = \underline{apr}_{R_v}(A) = (\overline{apr}_{R_u}(A^c))^c = \emptyset,$$
$$\overline{apr}_{R_u}(A) = \{1,2,3,4,5,6,7\} = (\underline{apr}_{R_u}(A^c))^c,$$
$$O_{R^c_u}(O_{R^c_u}(B)) = O_{R^c_u}(B) = Ib_{R^c}(A^c) = \{8,9,10\}.$$

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