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The Topological Centers Algebra of Banach Algebra and the Relations of Them

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Abstract

Let A be Banach algebra and A^*, A^{**} be the first and second dual space of its, respectively. In this paper, we show that for Banach algebra A if Z_1 and Z_2 are the topological center of A^{**} with respect to first and second Arens product, then we have $Z_1A \subseteq Z_1$. and $AZ_2 \subseteq Z_2$.. Specially for some condations the Banach algebra A is left strongly Arens irregular.

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Introduction

Let A be Banach algebra. A bounded net $(e_\alpha)_{\alpha\in I}$ in A is a bounded approximate identity ($= BAI$) if for each $a \in A$, $ae_\alpha \longrightarrow a$. For $a \in A$ and $f \in A^*$, we denote by $f.a$ and $a \diamond f$ respectively, the functionals on A defined by $< f.a, b >=< f, ab >$ and $< a \diamond f, b >=< f, ba >$. We denote by A^*A and AA^*, respectively, $\{f.a : \ a \in A \ and \ f \in A^*\}$, $\{a \diamond f : \ a \in A \ and \ f \in A^*\}$ of A^*.

In this paper, we show WSC, for weakly sequentially complete Banach algebra A and WCC, for weakly completly continuous that is a Banach algebra A is said to be WCC if for each $a \in A$, the multiplication operatore $x \to ax$ is weakly compact. We say that the Banach algebra A is unital if there exists an

element as $e \in A$ such that $e.x = x.e = x$ for each $x \in A$.
Let A be a Banach algebra with a BAI. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that A^* factor on the left (right). If both equality $A^*A = AA^* = A^*$ hold, then we say that A^* is two-sided factor.

Let $a, b \in A$ and $f \in A^*$ and $F, G \in A^{**}$ then the first Arens multiplication are defined as

$$< f.a, b >=< f, ab >$$

$$< F.f, a >=< F, f.a >$$

$$< F.G, f >=< F, Gf > .$$

Which is clear $F.f \in A^*$ and $F.G \in A^{**}$.
The second Arens product is defined as follows
For $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$, the element $a \diamond f$, $f \diamond F$ of A^* and $F \diamond G$ of A^{**} are defined by the equalities

$$< b, a \diamond f >=< ba, f >$$

$$< a, f \diamond F >=< a \diamond f, F >$$

$$< f, F \diamond G >=< f \diamond F, G > .$$

We recall the definition of mixed unite which play a fundamental role in this paper. An element E of A^* is said to be a mixed unit if E is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, E is a mixed unit if and only if , for each $F \in A^{**}$, $F.E = E \diamond F = F$. We say that A^{**} is unital with respect to first Arens product , if there exists an element $E \in A^{**}$ such that $F.E = E.F = F$ for all $F \in A^{**}$, and is unital with respect to second Arens product , if there exists an element $E \in A^{**}$ such that $F \diamond E = E \diamond F = F$ for all $F \in A^{**}$.

Theorem 1. An element E of A^{**} is said to be mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha \in I}$ in A.
Proof: see [11, p.146].

Lemma 2. Let A be a Banach algebra with a sequential BAI $(e_\alpha)_{\alpha \in I}$. Then, A^* factore on the left (right)if and only if for each $f \in A^*$, the nets $(f.e_\alpha)_{\alpha \in I}$, $(e_\alpha \diamond f)_{\alpha \in I}$ converges *weakly* to f.
Proof: See (2.1) from [2].

Theorem 3. Let A be a Banach algebra with a sequential BAI $(e_n)_{n\in N}$ such that A^* is WSC. Then A^* factors on the left if and only if A^* factors on the right.
Proof: See (2.9) from[2].

Theorem 4. Let A be a Banach algebra with BAI. Then the following assertion hold:
i)$A^*A = A^*$ if and only if the algebra $(A^{**}, .)$ is unital.
ii)$AA^* = A^*$ if and only if the algebra $(A^{**}, \diamond)$ is unital.
iii)$A^*A = AA^* = A^*$ if and only if the unite element of the algebra $(A^{**}, .)$ and $(A^{**}, \diamond)$ are the same.
For proof see Propostion (2.2) from [2].

Theorem 5. Let A have a BAI. Then $(A^*A)^\perp$ is the ideal of right annihilators in A^{**}.

Proof: If $\nu \in (A^*A)^\perp$ then $< \nu f, x >=< \nu, fx >= 0$ $(for\ all\ f \in A^*, x \in A)$, so that $< F\nu, f >=< F, \nu f >= 0$ $(for\ all\ F \in A^{**}, f \in A^*)$ and ν is a right annihilator.

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Definition 6. Suppose that $F, G \in A^{**}$ and $F.G, F \diamond G$ are the first (second) Arens Multiplication in A^{**}, respectively. Then the mapping $F \rightarrow F.G$, for G fixed in A^{**}, is $weak^* - to - weak^*$ continuous, but the mapping $F \rightarrow G.F$ for G fixed in A^{**} is in general not $weak^* - to - weak^*$ continuous. Whence, the first topological center of A^{**} with respect to first Arens product is defined as follows

$$Z_1 = \{G \in X^{**} :\ F \longrightarrow G.F\ is\ w^* - w^*\ continuous\ on\ A^{**}\}.$$

We know that $a \diamond f, f \diamond F \in A^*$ and $F \diamond G \in A^{**}$. For G fixed in A^{**} , the mapping $F \rightarrow G \diamond F$ is $weak^* - weak^*$ continouse on A^{**}, but the mapping $F \rightarrow F \diamond G$ is in generally not $weak^* - weak^*$ continuous on A^{**} unless $G \in A$. Whence, the second topological center of A^{**} with respect to second Arense product is defined as follows

$$Z_2 = \{G \in A^{**} :\ F \longrightarrow F \diamond G\ is\ w^* - w^* continuous\ on\ A^{**}\}.$$

It is clear $A \subseteq Z_1 \bigcap Z_2$ and Z_1, Z_2 are closed subalgebra of A^{**} endowed with the first (second) Arens Multiplication.

If ,for each $F, G \in A^{**}$, the equality $F.G = F \diamond G$ holds ,then the algebra A is said to be Arens regular.In this case $Z_1 = Z_2 = A^{**}$.
The other extrem situation is $Z_1 = A$, in which case A is called left strongly Arense irregular, see[5,6].
We recall that the topological center of A^{**} can defined to be the set of functional $F \in A^{**}$ which satisfy $F.G = F \diamond G$ for all $G \in A^{**}$. In other words, the topological centers of A^{**} with respect to first and second Arense products are defined as follow:

$$Z_1 = \{F \in A^{**} : \ F.G = F \diamond G \quad \forall G \in A^{**}\},$$

$$Z_2 = \{F \in A^{**} : \ G.F = G \diamond F \quad \forall G \in A^{**}\}$$

In above discussion, we refer that $A \subseteq Z_1 \cap Z_2$,i.e; for all $a \in A$, $f \in A^*$ and $F \in A^{**}$ we have
i) $f.a = f \diamond a$ or $a.f = a \diamond f$.
ii) $a.F = a \diamond F$ or $F.a = F \diamond a$.
Of course if $b \in A$, then we have

$$< f \diamond a, b >=< a, b \diamond f >=< ba, f >= f(ab).$$

Since $< f.a, b >= f(ab)$, we have $f.a = f \diamond a$.
Also we have

$$< a.F, f >=< a, F.f >=< F.f, a >=< F, f.a >=< F, f \diamond a >=< a \diamond F, f > .$$

Thus, we conclude that $a.F = a \diamond F$. Also for Banach algebra A we recall that

$$M_1 = \{F \in A^{**} \ : \ A.F \subseteq A\},$$

$$M_2 = \{F \in A^{**} \ : \ F \diamond A \subseteq A\}$$

Theorem 7. Let A be a Banach algebra with a BAI. Then
a) A^* factors on the left if and only if $M_1 \subseteq Z_1$
b) A^* factors on the right if and only if $M_2 \subseteq Z_2$

Proof: Let $A^*A = A^*$, and let $G \in M_1$ Since , for all $a \in A$, $a.G \in A$ and any f in A^* is the form $f = g.a$ for some $g \in A^*$ and $a \in A$. Then for all $f \in A^*$, we have

$$< G.F_\alpha, f >=< G.F_\alpha, g.a >=< a.(G.F_\alpha), g >$$

$$=< (a.G).F_\alpha, g >\longrightarrow< (a.G).F, g >=< G.F, f > .$$

Thus, $G \in Z_1$. It is similar that $M_2 \subseteq Z_2$.
Conversely, assume that $M_1 \subseteq Z_1$. Since A is BAI,by Theorem1 A^{**} has a mixed unit as E. Thus, we have $a.E = a$ implies that $E \in M_1 \subseteq Z_1$. Hence, for all $G \in A^{**}$, we have $E.G = E\diamond G = G$. Then A^{**} is unital so by Theorem4, A^* is factor on the left.

Corollary 8. Let A be a Banach algebra with a BAI. Then the following assertion are hold
i) the algebra $(A^{**}, .)$ is unital if and only if $M_1 \subseteq Z_1$.
ii) the algebra $(A^{**}, \diamond)$ is unital if and only if $M_2 \subseteq Z_2$.
Also, if A is WSC, then $M_1 \subseteq Z_1$ if and only if $M_2 \subseteq Z_2$ if and only if A is unital.
Proof: By above Theorem and Theorem4 and also by Theorem2.6 from [2], proof is clear.

Theorem 9. Let A be a Banach algebra. Then the following assertion are hold
i) $Z_1A \subseteq Z_1$.
ii) $AZ_2 \subseteq Z_2$.
Proof: Let $G \in Z_1$ and $a \in A$. Then the mapping $F \rightarrow F.G$ is $w^* - w^*-$continuous. Let $(F_\alpha)_{\alpha\in I} \subset A^{**}$ and $F_\alpha \longrightarrow F$ (w^*). We claim that $a.F_\alpha \longrightarrow a.F$ (w^*). Let $f \in A^*$. Then

$$< a.F_\alpha, f >=< a, F_\alpha.f >=< F_\alpha.f, a >=< F_\alpha, f.a >\rightarrow< F, f.a >$$

$$=< F.f, a >=< a, F.f >=< a.F, f > .$$

Thus, $a.F_\alpha \longrightarrow a.F$ (w^*).
Since $G \in Z_1$, $< G.(aF_\alpha), f >\rightarrow< G.(aF), f >$. Consequently, we have $(G.a)F_\alpha \rightarrow (G.a)F(w^*)$. Therefore, we conclude that $G.a \in Z_1$,and so $Z_1A \subseteq Z_1$. It is the same that $AZ_2 \subseteq Z_2$.

Definition 10. We recall that a cardinal number k is called (real valued) Measurable if for any set Γ with cardinality $|\Gamma| = k$, there exist a diffused probability measure on the power set $P(\Gamma)$. Now we introduce the following crucial concept which is a general property for Banach algebras.

Definition 11. Let A be a Banach algebra and let k be a cardinal number. We say that A^* has

i) the left A^{**} factorization property of level k [property (F_k), for short] if for any family of functionals $(h_\alpha)_{\alpha\in I} \subseteq Ball(A^*)$ with $|I| = k$, there exist a family $(t_\alpha)_{\alpha\in I} \subseteq Ball(A^{**})$ and one single functional $h \in A^*$ such that the factorization $h_\alpha = t_\alpha.h$ holds for all $\alpha \in I$.
ii) the left uniform A^{**} factorization property of level k [property (UF_k), for short] if there is a family $(t_\alpha)_{\alpha\in I} \subseteq Ball(A^{**})$ with $|I| = k$, such that for any family of functional $(h_\alpha)_{\alpha\in I} \subseteq Ball(A^*)$, there is one single functional $h \in A^*$ such that the factorization formula $h_\alpha = t_\alpha.h$ holds for all $\alpha \in I$.

Definition 12. let X be Banach space and $k \geq \aleph_0$ a cardinal number.
(i) A functional $F \in X^{**}$ is called $w^* - k - continuous$ if for all net $(f_\alpha)_{\alpha\in I} \subseteq Ball(A^*)$ of cardinality $\aleph_0 \leq |I| \leq k$ with $f_\alpha \to 0$ *weakly**, we have
$< F, f_\alpha > \to 0$.
(ii) We say that X has the Mazur property of level k [property (M_k), for short] if every $w^* - k - continuous$ functional $F \in X^{**}$ actually is an element of X. As is well-known a Banach space X is said to have the (classical) Mazur property if every $w^* - sequentially$ continuous functional $F \in X^{**}$ belong to X.

Theorem 13. Let A be a Banach algebra satisfying (M_k) and whose dual A^* has the property (F_k), for some $k \geq \aleph_0$. Then, the Banach algebra A is left strongly Arens irregular,i.e; $Z_1 = A$.

Proof: Let $(h_\alpha)_{\alpha\in I} \subseteq Ball(A^*)$, where $\mid I \mid \leq k$, which $h_\alpha \longrightarrow 0$ (w^*). By property (F_k), for all $\alpha \in I$, we have $h_\alpha = t_\alpha.h$ where $(t_\alpha)_{\alpha\in I} \subseteq Ball(A^{**})$ and $h \in A^*$. Since the net $(t_\alpha)_{\alpha\in I} \subseteq Ball(A^{**})$ is bounded, there exist a $w^* - convergent$ subnet $(t_{\alpha_\beta})_\beta$ such that $E = w^* - lim_\beta t_{\alpha_\beta} \in Ball(A^{**})$. since for all $a \in A$:

$$< E.h, a >=< E, h.a >= lim_\beta < t_{\alpha_\beta}, h.a >= lim_\beta < t_{\alpha_\beta}.h, a >$$
$$= lim_\beta < h_{\alpha_\beta}, a >= 0.$$

Thus, $E.h = 0$. Now let $F \in Z_1$. Then we have

$$lim_\alpha < F, h_\alpha >= lim_\beta < F, h_{\alpha_\beta} >= lim_\beta < F, t_{\alpha_\beta}.h >$$
$$= lim_\beta < F.t_{\alpha_\beta}, h >=< F.E, h >=< F, E.h >= 0.$$

Since A has the Mazur property, we conclude that $F \in A$.

Corollary 14. For locally compact non-compact group G with non-measurable cardinality, the Banach algebra $M(G)$ is left strongly Arens irregular,i.e, $Z_1(M(G)^{**}) = M(G)$.
For detail see [5].
Also for locally compact group G, $L^1(G)$ is strongly Arens irregular, that is, $Z_1(L^1(G)^{**}) = L^1(G)$. For detail see [14].

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